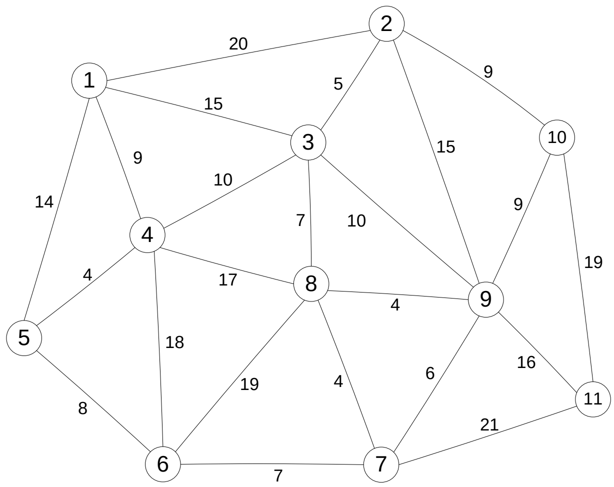
2.1 Minimum-cost spanning tree

Find the minimum-cost spanning tree in the graph given in the figure by using Prim's algorithm, starting from the node 3.



2.2 Kruskal's algorithm

In 1956 Joseph Kruskal proposed the following greedy algorithm to find a minimum-cost spanning tree in an arbitrary connected undirected graph $G = (N, E)$ with a cost c_e attached to each edge $e \in E$.

1) Sort the edges of E as $\{e_1, \ldots, e_m\}$ where $c_{e_1} \leq c_{e_2} \leq \ldots \leq c_{e_m}$
2) Let $i = 1$ and initialize the subgraph $G' = (N, F)$ of G with $F = \emptyset$ (G' consists of n connected components[1] corresponding to the isolated nodes)
3) WHILE $|F| < n - 1$ DO

   IF the two endpoints of the edge e_i belong to different connected components of the current subgraph G' THEN $F := F \cup \{e_i\}$ and merge the two connected components

   $i := i + 1$

   END
4) Return the spanning tree $G' = (N, F)$

In other words, we order the edges by increasing (non-decreasing) cost, we consider the edges in that order and, at each step, we select the current edge (which is one of the cheapest edges still available) only if it does not create a cycle with the previously selected edges. The algorithm terminates when $n - 1$ edges have been selected.

[1]A connected component of an undirected graph is a subgraph in which any two nodes are connected, and which is connected to no other nodes.

a) Describe an efficient way to identify/keep track of the connected components of the subgraph G' and to check that a new edge is creating a cycle with the previously selected edges (is connecting two distinct connected components of G').

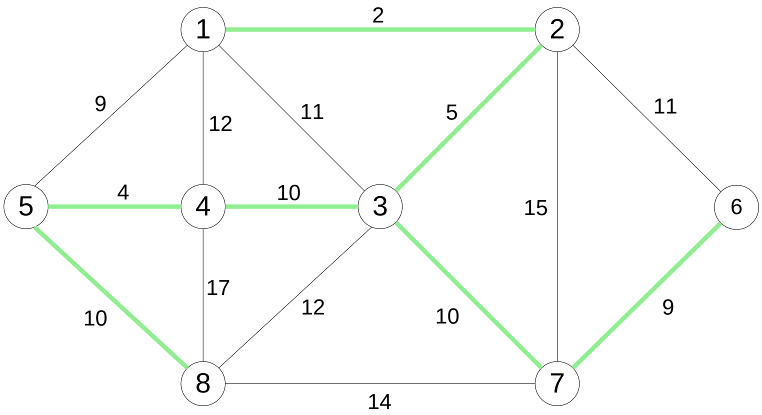
b) Determine the overall computational complexity of this simple implementation of Kruskal's algorithm.

c) By invoking the optimality condition for minimum-cost spanning trees, verify that Kruskal's algorithm is exact, i.e., is guaranteed to provide an optimal solution for any undirected graph with costs on the edges.

d) Find the maximum-cost spanning tree in the graph of the previous exercise by using a straightforward adaptation of Kruskal's algorithm.

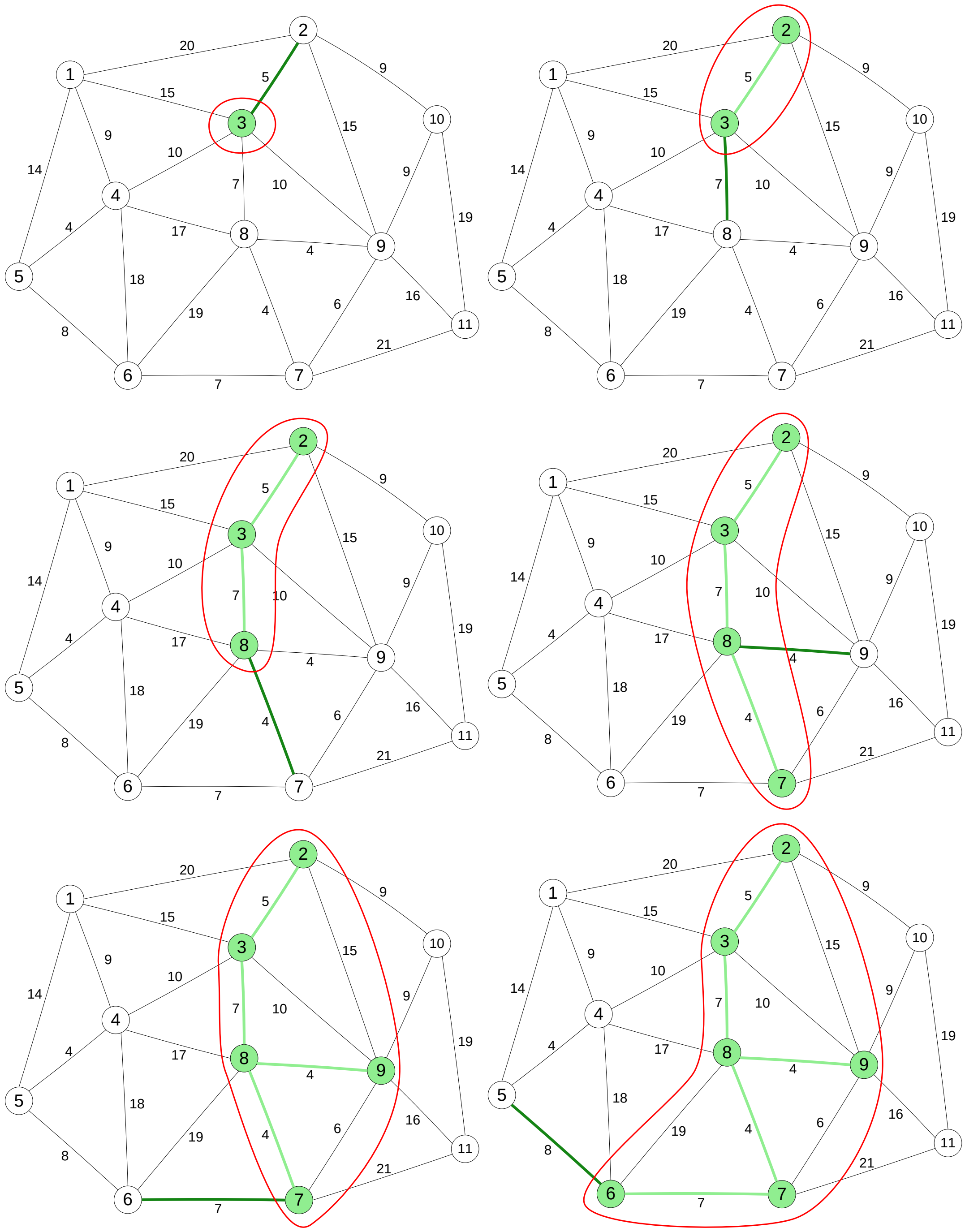
2.3 Optimality check

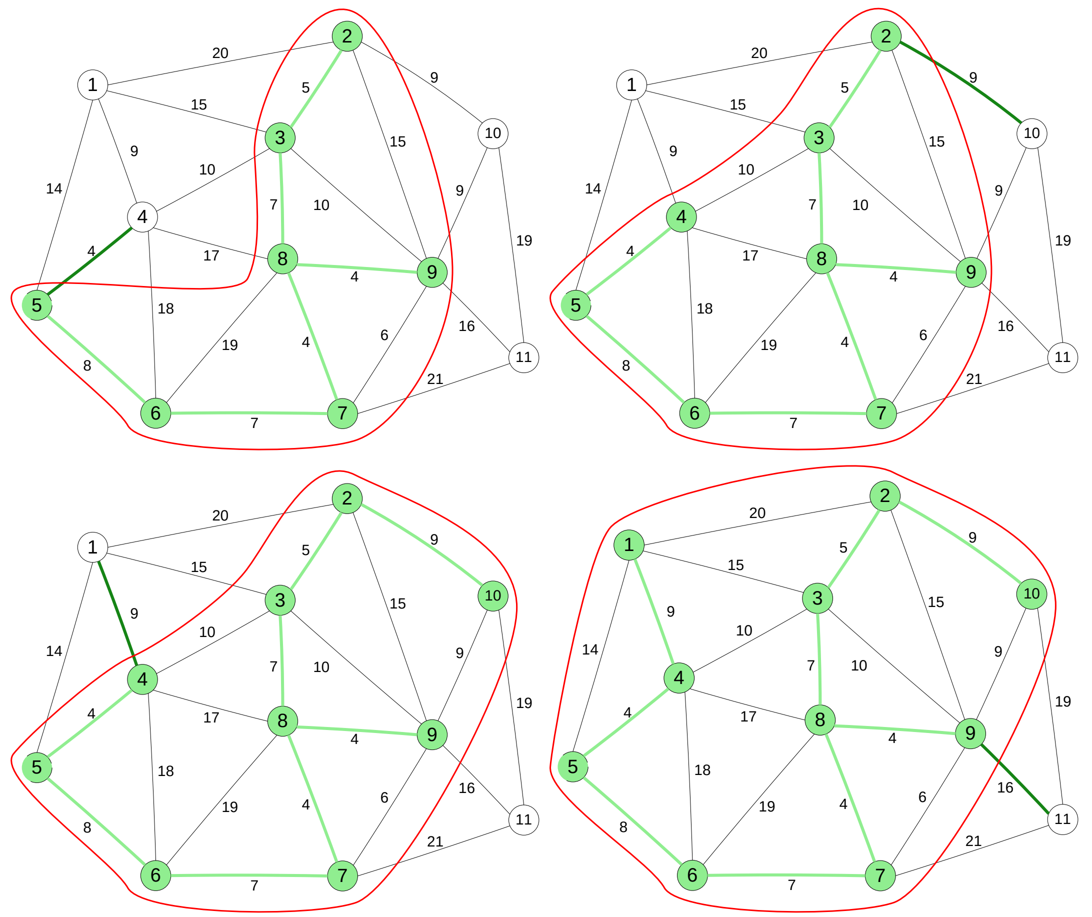
Without applying any one of Prim's and Kruskal's algorithms, verify whether the following spanning tree is of minimum cost.



SOLUTION

2.1 **Minimum-cost spanning tree**. We apply Prim's algorithm, starting at node 3. At each iteration, the set S of the nodes in which the edges selected so far are incident is highlighted in red. The partial spanning tree is highlighted in light green. Among all edges in the cut $\delta(S)$, induced by S, the edge, among those of minimum cost, that is going to be added to the partial spanning tree is highlighted in dark green.

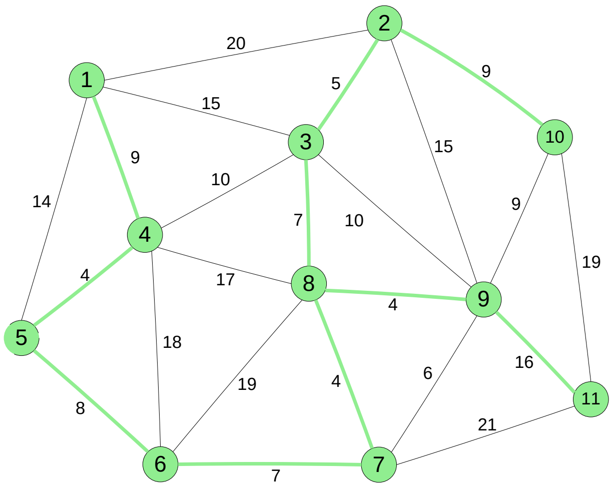




The edges are added in the following order

reached nodes	edge	cost	iteration
3	(2,3)	5	1
2,3	(3,8)	7	2
2,3,8	(7,8)	4	3
2,3,7,8	(8,9)	4	4
2,3,7,8,9	(6,7)	7	5
2,3,6,7,8,9	(5,6)	8	6
2,3,5,6,7,8,9	(4,5)	4	7
2,3,4,5,6,7,8,9	(2,10)	9	8
2,3,4,5,6,7,8,9,10	(1,4)	9	9
1,2,3,4,5,6,7,8,9,10	(9,11)	16	10=n-1 $\to$ HALT

Since $S = V$, i.e., every node has been reached, the algorithm halts. The minimum-cost spanning tree that has been found has total cost 73. It is shown in the following figure.



2.2 **Kruskal's algorithm**.

a) To identify/keep track of the connected components (subtrees) of the subgraph G', we use a vector v with as many components as vertices in the graph, where $v[i]$ indicates the index of the connected component containing node i. At the beginning of the algorithm, we start with $v[i] = i$ for $i = 1, \ldots, n$. When an edge $e = \{i, j\}$ is considered for addition to G', we compare the values $v[i]$ and $v[j]$. If $v[i] \neq v[j]$, then we can add the edge e to G' because it does not create a cycle. Since the two connected components of indices $v[i]$ and $v[j]$ are merged, the indices are updated as follows: in the vector v we substitute each occurrence of the index of node i with that of node j. If $v[i] = v[j]$, edge e is skipped because it would create a cycle.

b) The m edges can be ordered by non-decreasing cost in $O(m \log m)$, which is $O(m \log n)$ since $m \log m \leq m \log n^2 = 2m \log n$. At most m edges are considered for addition to the current subgraph G'. At each iteration, an edge $e = \{i, j\}$ is considered and the vector v is updated (in $O(n)$) only if $v[i] \neq v[j]$. Since a merging operation occur exactly $n - 1$ times (a spanning tree contains $n - 1$ edges), the overall complexity is $O(m \log n + m + n^2) = O(m \log n + n^2)$.

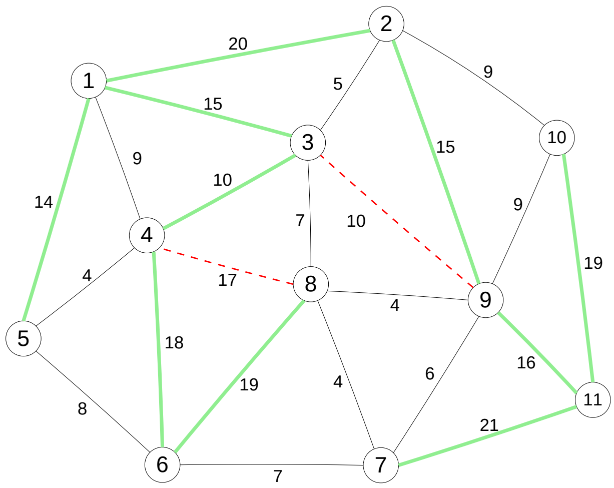
c) To verify that Kruskal's algorithm is exact, we just need to recall that the edges are considered in order of non-decreasing cost and to invoke the optimality condition for minimum-cost spanning tree. Since each edge e that has been discarded (not added to F) has a cost c_e which is at least as large as the cost of all the previously selected edges, it is not a cost-decreasing edge, i.e., $c_e \geq c_f$ for every edge f in the unique cycle that would have been created if e was added to F. According to the optimality condition for minimum-cost spanning trees, the resulting spanning tree (with edge set F) is of minimum total cost because no cost-decreasing edge exists.

d) To determine a maximum-cost spanning tree in the given undirected graph $G = (V, E)$, we sort all the edges of G by nonincreasing cost and consider them one by one in that order. Let e be the edge considered at the current iteration and F be the set of edges selected so far. If adding e to F creates a cycle, e is dropped. Otherwise, e is added to F, and a new iteration is performed. The algorithm terminates when $n - 1$ edges have been selected, namely, when $|F| = n - 1$.

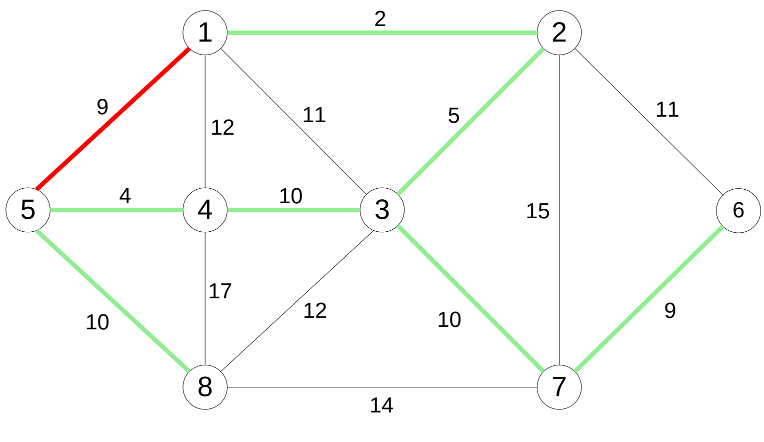
The edges are considered in the following order

connected components	edge	cost	iteration
	(7,11)	21	1
$\{7,11\}$	(1,2)	20	2
$\{1,2\},\{7,11\}$	(6,8)	19	3
$\{1,2\},\{7,11\},\{6,8\}$	(10,11)	19	4
$\{1,2\},\{7,10,11\},\{6,8\}$	(4,6)	18	5
$\{1,2\},\{7,10,11\},\{4,6,8\}$	(4,8)	NO	(introduces a cycle)
$\{1,2\},\{7,10,11\},\{4,6,8\}$	(9,11)	16	6
$\{1,2\},\{7,9,10,11\},\{4,6,8\}$	(1,3)	15	7
$\{1,2,3\},\{7,9,10,11\},\{4,6,8\}$	(2,9)	15	8
$\{1,2,3,7,9,10,11\},\{4,6,8\}$	(1,5)	14	9
$\{1,2,3,5,7,9,10,11\},\{4,6,8\}$	(3,9)	NO	(introduces a cycle)
$\{1,2,3,4,5,6,7,8,9,10,11\}$	(3,4)	10	10=n-1 $\rightarrow$ HALT

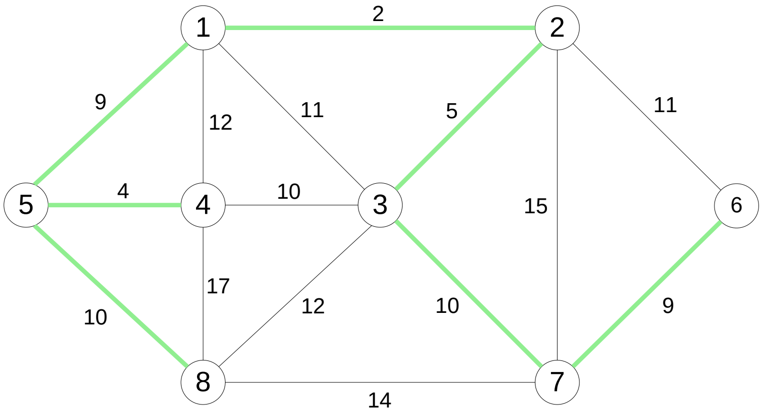
A spanning tree of maximum cost, of value 167, is shown in green in the following figure. The dropped edges are highlighted in red.



2.3 **Optimality proofs**. It suffices to verify that there exists a diminishing edge. By inspection, we observe that, by adding edge $(1,5)$ to the tree, the cycle $(1,5,4,3,2,1)$ is introduced. In such cycle, edge $(4,3)$ has a strictly larger cost than $(1,5)$.



Therefore, by removing edge $(4,3)$ and adding edge $(1,5)$, a spanning tree of strictly smaller total cost is obtained. It is shown in the following figure.


1
2
3
4
5
6
7
8
2
9
12
11
5
11
4
10
15
17
12
10
10
9
14