Foundations of Operations Research

Master of Science in Computer Engineering

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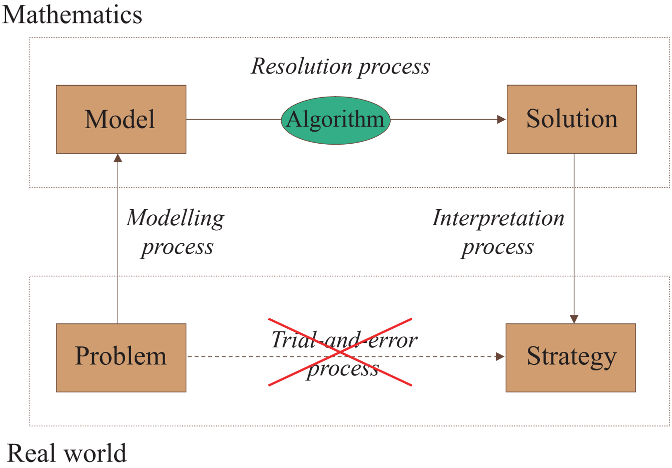
Lesson 2: Modelling decision-making problems Como, Fall 2013

The modelling approach



Build a mathematical equivalent of the problem and solve it

- faster
- cheaper
- safer
- more adaptable
- more informative
- . . .

Basic ingredients of decision-making models

A model is a simplified representation of the real world composed of mathematical objects (numbers, sets, functions,. . .) which correspond to

- the relevant elements of the problem
- the relevant relationships among them

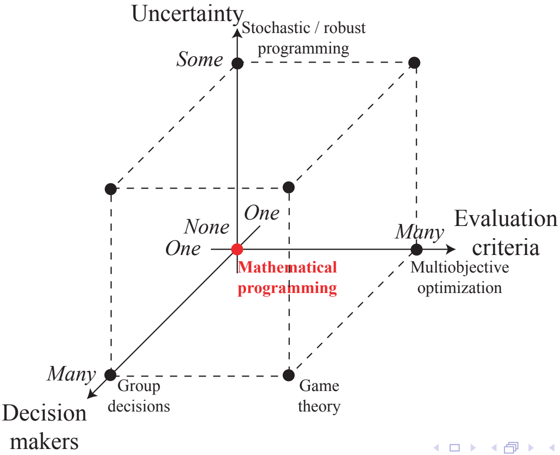
In decision-making models, we must define

- decision variables: what quantities can be decided?
- feasible solution set: what decisions are acceptable?
- evaluation criteria: what quantites measure how good a decision is?

A taxonomy of decision-making problems

Decision-making problems can be classified based on three features

1. the number of decision makers: one or many?
2. the number of criteria used to evaluate solutions: one or many?
3. the level of uncertainty of the result: deterministic, stochastic, uncertain?



Mathematical Programming

We focus on the specific case of Mathematical Programming

1. a single decision maker
2. a single evaluation criterion
3. fully deterministic environment

Under these hypotheses, decision-making problems can be represented as

$$\begin{array}{l}\mathrm{opt}\ f(x)\\ x \in X\end{array}$$

$$\begin{array}{ll} \mathrm{opt}\ f(x) \\ x \in X \end{array} \qquad \longrightarrow \qquad \begin{array}{ll} \min\ f(x) & \\ g_i(x) \leq 0 & i = 1, \dots, m \end{array}$$

where

- each solution $x \in \mathbb{R}^n$ is a vector of n decision variables x_j (real numbers, with $j = 1, \dots, n$); each x_j is a quantity that the decision maker can fix (to some extent)
- the feasible region $X \subseteq \mathbb{R}^n$ is a set of vectors, which discriminates the feasible solutions from the unfeasible ones; usually, it is defined by inequalities $g_i(x) \leq 0$ named constraints

$$X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \dots, m\}$$

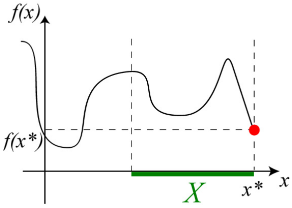
- the objective function $f : X \to \mathbb{R}$ is a quantitative measure of the quality of each feasible solution (opt usually stands for min or max)

Why $g_i \leq 0$? Are min *and* max *problems different?*

Global optima

Solving a mathematical programming problem amounts to finding a globally optimal solution, i. e. a solution x^* such that

$$\begin{cases} x^* \in X \\ f(x^*) \leq f(x) \qquad \forall x \in X \end{cases}$$



Possible cases:

1. the problem has a unique optimal solution
2. the problem has several equivalent optimal solutions (possibly even infinite solutions)
3. the problem is unbounded: $\forall c \in \mathbb{R}, \exists x_c \in X$ such that $f(x_c) \leq c$
4. the problem is unfeasible: $X = \emptyset$

Special cases of mathematical programming

1. Linear Programming (LP): $f(x)$ and all $g_i(x)$ are affine functions

$$f(x) = c_1x_1 + \ldots + c_nx_n + d$$
$$g_i(x) = a_{i1}x_1 + \ldots + a_{in}x_n + b_i \qquad i = 1, \ldots, m$$

   where $c_j, d, a_{ij}, b_i \in \mathbb{R}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

2. Integer Programming (LP): $x_j \in \mathbb{Z}$ for $j = 1, \ldots, n$
   Equivalently, $\sin(\pi x_j) = 0$

3. Binary or $0-1$ Programming (BP): $x_j \in \{0, 1\}$ for $j = 1, \ldots, n$
   Equivalently, $x_j(1 - x_j) = 0$

4. Combinatorial Optimization (CO): X represents the collection of all subsets of a given ground set G which enjoy some desired properties
   $x_j = 1$ usually means that element $j \in G$ belongs to the solution,
   $x_j = 0$ that it does not

Practical advice

Do not try to solve the problem: limit yourselves to describing it

1. Define clearly the objective:
   how to compute it and its unit of measure

2. Define clearly one by one the quantities that make a solution unfeasible: how to compute them and their units of measure

   Describe what is forbidden, not what is allowed!

3. Choose the decision variables: what quantities have
   - a value which can be fixed (to some extent)?
   - an influence on the objective function?
   - an influence on the feasibility of the solution?

4. Express the objective through the decision variables

5. Express the constraints through the decision variables

If you fail in step 4 or 5

- go back to step 3 and change the decision variables
- go back to steps 1 and 2 and define more clearly the problem

Example: production mix (statement)

Your customer produces three types of electronic devices, and all of them pass through three main phases: assembly, refinement and quality control.

The materials required to build the devices are available in abundance, as well as energy and the space to keep materials and final products.

Labour is a limiting resource: the workforce is fixed and cannot be exchanged among the three phases. During the planning horizon, the customer has 300 000 *seconds of work for assembly,* 250 000 *for refinement and* 180 000 *for quality control.*

Each device requires a known time for each of the three phases, as reported in the following table:

	D_1	D_2	D_3
Assembly	80	70	30
Refinement	70	90	20
Quality control	40	30	20

The company is sure to sell any produced quantity of the three devices, and knows exactly the profit made out of each device of the three types: 1.6 *euros for the first device,* 1 *euro for the second,* 2 *euros for the third one.*

The company aims to make the maximum possible profit in this situation.

Example: production mix (analysis)

- The objective is profit, expressed in euros, to be maximized
- Unfeasibility does not derive from lack of materials, energy, space
- Unfeasibility derives from lack of time:
    - not enough time for assembly, expressed in seconds
    - not enough time for refinement, expressed in seconds
    - not enough time for quality control, expressed in seconds
- Producing negative quantities is unfeasible
  (*obvious? there is nothing "obvious" in modelling*!)
- Producing without consuming the specified time is unfeasible

Example: production mix (analysis)

Decision variables (some alternative suggestions)

- the total time required to produce each device (in seconds)
  (*how do you compute the objective and the feasibility?*)
- the total time consumed by each phase (in seconds)
  (*ok for feasibility, but how do you compute the objective?*)
- the total time devoted to each phase for each device (in seconds)
- the quantities of each device produced (number of devices)

The last two are both acceptable: let us consider the latter:

- x_1 = number of devices of type 1 produced during the time horizon
- x_2 = number of devices of type 2 produced during the time horizon
- x_3 = number of devices of type 3 produced during the time horizon

Example: production mix (model)

- Objective function: the total profit is the sum over the three types of devices of the profit provided by each type multiplied by the number of devices of that type

$$\max f(x) = 1.6\, x_1 + 1\, x_2 + 2\, x_3$$

- Constraint: the total time required for assembly, refinement and control quality is the sum over the three types of devices of the time required for each device of that type multiplied by the number of devices; they are all bounded from above

$$\begin{aligned} 80x_1 + 70x_2 + 30x_3 &\leq 300\,000 \\ 70x_1 + 90x_2 + 20x_3 &\leq 250\,000 \\ 40x_1 + 30x_2 + 20x_3 &\leq 180\,000 \end{aligned}$$

- Constraints: the quantities produced are nonnegative

$$x_1 \geq 0 \qquad x_2 \geq 0 \qquad x_3 \geq 0$$

Example: another production mix (statement)

Your customer produces small and large chess boards.

Both are made of the same kind of wood, of which 200 *kg per week are available; each small board requires* 1 *kg, each large board* 3 *kg.*

Both are made on the same machine, which requires 3 *hours to produce a small board,* 2 *hours to produce a large one, and which works* 160 *hours a week.*

The company is sure to sell any number of produced boards, with a profit of 5 *euros for the small boards,* 20 *euros for the large ones.*

The company aims to make the maximum possible profit per week.

Try and do it yourself...

Example: another production mix (model)

Decision variables:

- x_s = number of small boards produced per week
- x_ℓ = number of large boards produced per week

$$
\begin{aligned}
\max f = 5x_s + 20x_\ell \\
1\ x_s + 3\ x_\ell &\leq 200 \\
3\ x_s + 2\ x_\ell &\leq 160 \\
x_s, x_\ell &\geq 0
\end{aligned}
$$

Should we require $x_s, x_\ell \in \mathbb{N}$?

The importance of being abstract

Many different problems have the same structure, and therefore can be

- described in a similar way
- solved in a similar way

... or in the same way?

In production mix problems, we have:

- a set of products P (devices, boards ...)
- a set of resources R (time, materials, energy, workers ...)
- a unitary profit p_j for each product $j \in P$
- an availabile amount r_i for each resource $i \in R$
- a consumption a_{ij} for each resource $i \in R$ and each product $j \in P$

The consequence is a general model for the production mix problem:

$$\max f = \sum_{j \in P} p_j \, x_j$$

$$\sum_{j \in P} a_{ij} \, x_j \leq r_i \qquad i \in R$$

$$x_j \geq 0 \qquad j \in P$$

Example: the knapsack (statement)

Your customer is an international jewel thief, who has planned in any detail a "job" in a famous jewellery.

The jewels will be taken away in a perfectly elastic bag: their volume is not a problem. Their weight, however, is limiting: the thief can carry at most 600 *g, to be safe during the escape. The weight of each jewel is known exactly, and the total weight is too large to take them all.*

Jewel	A	B	C	D	E	F	G	H
Weight	150	150	60	100	125	100	50	80

The value of each jewel in thousands of euros is known exactly, as well.

Jewel	A	B	C	D	E	F	G	H
Value	11	9	13	10	8	7	3	5

The thief aims to steal safely the worthiest pack of jewels.

Example: the knapsack (analysis)

- The objective is the total value stolen, expressed in Keuros, to be maximized
- Unfeasibility does not derive from the volume or shape of the jewels
- Unfeasibility derives from the total weight, possibly exceeding the thresold (both must be expressed in kg)
- Jewels can be stolen or not (*tertium non datur*)

Example: the knapsack (analysis)

How to represent "being stolen" and "being not stolen" for each jewel?

- the total value is the sum of the values of the stolen jewels
- the total weight is the sum of the weights of the stolen jewels

What decision variables should be adopted?

The usual trick is to use a binary variable for each jewel $j \in \{A, \dots, H\}$:

- $x_j = 1$ means that jewel j is stolen
- $x_j = 0$ means that jewel j is not stolen

(*They can be fixed, they influence the objective and the feasibility...*)

Example: the knapsack (model)

- Objective function: the total value stolen is the sum over all jewels of the value of each jewel times the corresponding binary variable

$$\max f(x) = 11x_A + 9x_B + 13x_C + 10x_D + 8x_E + 7x_F + 3x_G + 5x_H$$

- Constraint: the total weight stolen is the sum over all jewels of the weight of each jewel times the corresponding binary variable; it is bounded from above

$$150x_A + 150x_B + 60x_C + 100x_D + 125x_E + 100x_F + 50x_G + 80x_H \leq 600$$

- Constraints: the decision variables are binary

$$x_A, \ldots, x_H \in \{0, 1\}$$

Example: the knapsack (general problem)

Data

- a set of objects J
- a value v_j for each object $j \in J$
- a weight w_j for each object $j \in J$
- a capacity W

Decision variables

- $x_j = 1$ if object $j \in J$ belongs to the solution,
  $x_j = 0$ otherwise

$$\max f = \sum_{j \in J} v_j \, x_j$$

$$\sum_{j \in J} w_j \, x_j \leq W$$

$$x_j \in \{0, 1\}$$

Example: portfolio selection (statement)

An insurance company must select the investments to perform out of a given set of possible assets (stocks, bonds, etc. . .).

The total budget is 600 *Keuros, and the capital required by each asset is:*

Asset	A	B	C	D	E	F	G	H
Capital	150	150	60	100	125	100	50	80

Each asset refers to a given sector and nation, and has an expected return on investment (ROI):

Asset	A	B	C	D	E	F	G	H
Nation	Germany	Italy	USA	Italy	Italy	France	Italy	UK
Sector	autom.	autom.	ICT	ICT	real estate	real estate	short bond	long bond
ROI	11%	9%	13%	10%	8%	7%	3%	5%

The internal rules impose:

- *to select at most* 5 *investments (to avoid excessive fragmentation)*
- *to select at most* 3 *investments in Italy, at most* 3 *abroad (for geographical diversification)*
- *to select a bond if an ICT investement is selected (to limit risk)*

The company aims to maximize the expected ROI, respecting the rules

Example: portfolio selection (analysis)

- The objective is the total ROI, expressed in Keuros, to be maximized
- Unfeasibility derives from
  - exceeding the available capital
  - violating the rule on fragmentation
  - violating the rule on geographical diversification,
    either on Italian investments or on investments made abroad
  - violating the rule on risk (an ICT investment and no bond)
- Investments can be performed or not (no "partial" investment)

Binary decision variables:

- $x_j = 1$ if investment $j \in \{1, \ldots, 8\}$ is performed
- $x_j = 0$ otherwise

Example: portfolio selection (model)

- Objective function: slightly different from the knapsack (*notice the units!*)

$$\max f(x) = 0.11 \cdot 150x_A + 0.09 \cdot 150x_B + 0.13 \cdot 60x_C + 0.10 \cdot 100x_D + 0.08 \cdot 125x_E + 0.07 \cdot 100x_F + 0.03 \cdot 50x_G + 0.05 \cdot 80x_H$$

- Budget constraint: the same as the knapsack (*the power of modelling!*)

$$150x_A + 150x_B + 60x_C + 100x_D + 125x_E + 100x_F + 50x_G + 80x_H \leq 600$$

- Constraints: at most 5 investments performed

$$x_A + x_B + x_C + x_D + x_E + x_F + x_G + x_H \leq 5$$

- Constraints: at most 3 investments performed in Italy

$$x_B + x_D + x_E + x_G \leq 3$$

- Constraints: at most 3 investments performed abroad

$$x_A + x_C + x_F + x_H \leq 3$$

- Constraints: an ICT investment implies a bond investment

$$x_C \leq x_G + x_H \qquad x_D \leq x_G + x_H$$

- Constraints: the decision variables are binary

$$x_A, \ldots, x_H \in \{0, 1\}$$

Example: assignment (statement)

The production process in a factory requires 3 parallel independent tasks.

Each task requires exactly one machine, and each machine can perform at most one task at a time. To save time, the tasks are performed in parallel on different machines.

The cost in euros required to perform each task on each machine is reported in the following table:

	M_1	M_2	M_3
T_1	2	6	3
T_2	8	4	9
T_3	5	7	8

The factory managers want to assign the tasks to the machines so as to minimize the total cost.

Example: assignment (analysis)

- The objective is the total cost, expressed in euros, to be minimized
- Unfeasibility derives from
  - assigning less or more than one machine to a task
  - assigning more than one task to a machine (*in this case, also less*)
- Assignments must be done or not:
  an assignment involves a machine i and a task j;
  it is a pair (i, j) with $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$

Binary decision variables:

- $x_{i,j} = 1$ if task $i \in \{1, 2, 3\}$ is assigned to machine $j \in \{1, 2, 3\}$
- $x_{i,j} = 0$ otherwise

Example: assignment (model)

- Objective function: the total cost is the sum over all assignments of the cost of the assignment times the corresponding binary variable

$$\begin{aligned}\max f(x) = {} & 2x_{1,1} + 6x_{1,2} + 3x_{1,3} + \\ & 8x_{2,1} + 4x_{2,2} + 9x_{2,3} + \\ & 5x_{3,1} + 7x_{3,2} + 8x_{3,3}\end{aligned}$$

- Constraints: the number of machines assigned to each task is the sum of the variables associated to the task, and must be equal to 1

$$x_{1,1} + x_{1,2} + x_{1,3} = 1$$
$$x_{2,1} + x_{2,2} + x_{2,3} = 1$$
$$x_{3,1} + x_{3,2} + x_{3,3} = 1$$

- Constraints: the number of tasks assigned to each machine is the sum of the variables associated to the machine, and must be equal to 1

$$x_{1,1} + x_{2,1} + x_{3,1} = 1$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 1$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 1$$

- Constraints: the decision variables are binary

$$x_{i,j} \in \{0,1\} \text{ for } i,j \in \{1,2,3\}$$

Example: assignment (general problem)

Data

- a set of objects I
- a set of receivers J (with $|J| = |I|$)
- an assignment cost c_{ij} for each pair $(i, j) \in I \times J$

Decision variables

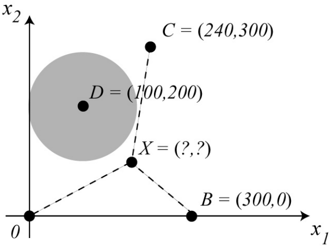
- $x_{i,j} = 1$ if objects $i \in I$ is assigned to receiver $j \in J$
- $x_{i,j} = 0$ otherwise

$$
\begin{aligned}
\max f = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{i,j} & \\
\sum_{j \in J} x_{i,j} = 1 & \qquad i \in I \\
\sum_{i \in I} x_{i,j} = 1 & \qquad j \in J \\
x_{i,j} \in \{0, 1\} & \qquad i \in I, j \in J
\end{aligned}
$$

Example: facility location (statement)

Oil is extracted from three pits, located in positions $A = (0, 0)$, $B = (300, 0)$ *and* $C = (240, 300)$.

A refinery must be built and connected to the three pits
The refinery must be at least 100 *km away from position* $D = (100, 200)$
The oil pipelines can be built everywhere



Decide the location of the refinery, so as to minimize the total pipeline cost, which is proportional its length.

Example: facility location (analysis)

- The objective is the total distance from the refinery to the three pits, expressed in metres, to be minimized
- Unfeasibility derives from locating the refinery not far enough from position D
- Unfeasibility does not derive from the pipelines
- The pipelines will be rectilinear

Decision variables:

- x_1 and x_2: cartesian coordinates of the refinery

Example: facility location (model)

- Objective function: the total distance of the refinery from the pits is the sum of the distances from each of the three pits

$$\min f(x) = \sqrt{\left[(x_1 - 0)^2 + (x_2 - 0)^2\right]} + \sqrt{\left[(x_1 - 300)^2 + (x_2 - 0)^2\right]} + \sqrt{\left[(x_1 - 240)^2 + (x_2 - 300)^2\right]}$$

- Constraint: the distance from position D must be larger than 100

$$\sqrt{(x_1 - 100)^2 + (x_2 - 200)^2} \geq 100$$

Do it yourself: diet problem

You go to the market to buy food for today; you can buy bread, milk, eggs, meat and dessert.

Bread costs 2 *euros/hg, milk* 3 *euros/hg, eggs* 4 *euros/hg, meat* 19 *euros/hg and dessert* 20 *euros/hg.*

You want to gain a minimum amount of nutrients from the food you buy: at least 2 000 *calories,* 50 *g of proteins and* 700 *mg of calcium.*

The following table reports the amount of nutrients for each available food (phantasy values, of course...):

	bread	milk	eggs	meat	dessert
calories/hg	110	160	180	260	420
proteins (g/hg)	4	8	13	14	4
calcium (mg/hg)	2	285	54	80	22

Determine the amounts of food to buy so as to minimize the total cost, while respecting the nutrition requirement.

Do it yourself: multi-period production

A factory produces a single product, at a unitary production cost which changes month after month as follows:

	Jan	Feb	Mar	Apr	May	Jun
Prod. cost (euros/unit)	20	25	30	40	50	60

The factory can produce a different maximum amount in each month:

	Jan	Feb	Mar	Apr	May	Jun
Prod. capacity (units)	1 500	2 000	2 200	3 000	2 700	2 500

The products are sold in blocks at the end of each month;
the demand and the price are known in advance:

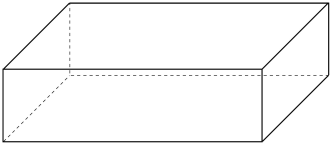
	Jan	Feb	Mar	Apr	May	Jun
Demand (units)	1 100	1 500	1 800	1 600	2 300	2 500
Price (euros/unit)	180	180	250	270	300	320

The products are first stored in a temporary depot for free; at the end of the month, the unsold ones are moved to a larger depot which can store 3 000 *units at* 2 *euros/unit per month.*

Determine the production plan in order to maximize the total profit (total revenue minus total production and storage costs).

Do it yourself: car design

You must design a new concept car, named Spatium, which will be shaped as a rectangular parallelepiped



The cost of the car has already been decided; it implies that the total surface of the car must be lower than $37m^2$.

The length of the car must be at least $4m$.

The width of the car must be between $1.5m$ *and* $2.5m$.

The height of the car must be at most equal to the width and at most $1/3$ *of the length.*

You want to maximize the volume of the car, expressed in litres.