Foundations of Operations Research

Solutions to the modelling exercises of Lesson 2

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Model 1: production mix

Statement: *Your customer produces three types of electronic devices, and all of them pass through three main phases: assembly, refinement and quality control.*

The materials required to build the devices are available in abundance, as well as energy and the space to keep materials and final products.

Labour is a limiting resource: the workforce is fixed and cannot be exchanged among the three phases. During the planning horizon, the customer has 300 000 *seconds of work for assembly,* 250 000 *for refinement and* 180 000 *for quality control.*

Each device requires a known time for each of the three phases, as reported in the following table:

	D_1	D_2	D_3
Assembly	80	70	30
Refinement	70	90	20
Quality control	40	30	20

The company is sure to sell any produced quantity of the three devices, and knows exactly the profit made out of each device of the three types: 1.6 *euros for the first device,* 1 *euro for the second,* 2 *euros for the third one.*

The company aims to make the maximum possible profit in this situation.

Model Decision variables:

- x_1 = number of devices produced of type 1 during the time horizon
- x_2 = number of devices produced of type 2 during the time horizon
- x_3 = number of devices produced of type 3 during the time horizon

$$\max f(x) = 1.6\, x_1 + 1\, x_2 + 2\, x_3 \tag{1a}$$
$$80x_1 + 70x_2 + 30x_3 \leq 300\,000 \tag{1b}$$
$$70x_1 + 90x_2 + 20x_3 \leq 250\,000 \tag{1c}$$
$$40x_1 + 30x_2 + 20x_3 \leq 180\,000 \tag{1d}$$
$$x_1, x_2, x_3 \geq 0 \tag{1e}$$

Model 2: another production mix

Statement *Your customer produces large and small chess boards.*

Both are made of the same kind of wood, of which 200 *kg per week are available; each small board requires* 1 *kg, each large board* 3 *kg.*

Both are made on the same machine, which requires 3 *hours to produce a small board,* 2 *hours to produce a large one, and which works* 160 *hours a week.*

The company is sure to sell any number of produced boards, with a profit of 5 *euros for the small boards,* 20 *euros for the large ones.*

The company aims to make the maximum possible profit per week.

Model

- x_s = number of small boards produced per week
- x_l = number of large boards produced per week

$$\max f = 5x_s + 20x_l \tag{2a}$$

$$1\ x_s + 3\ x_l \leq 200 \tag{2b}$$

$$3\ x_s + 2\ x_l \leq 160 \tag{2c}$$

$$x_s, x_l \geq 0 \tag{2d}$$

Model 3: knapsack

Statement *Your customer is an international jewel thief, who has planned in any detail a "job" in a famous jewellery.*

The jewels will be taken away in a perfectly elastic bag: their volume is not a problem. Their weight, however, is limiting: the thief can carry at most 600 *g, to be safe during the escape. The weight of each jewel is known exactly, and the total weight is too large to take them all.*

Jewel	A	B	C	D	E	F	G	H
Weight	150	150	60	100	125	100	50	80

The value of each jewel in thousands of euros is known exactly, as well.

Jewel	A	B	C	D	E	F	G	H
Value	11	9	13	10	8	7	3	5

The thief aims to steal safely the worthiest pack of jewels.

Model

- $x_j = 1$ means that jewel j is stolen
- $x_j = 0$ means that jewel j is not stolen

$$\max f(x) = 11x_A + 9x_B + 13x_C + 10x_D + \tag{3a}$$
$$+8x_E + 7x_F + 3x_G + 5x_H \tag{3b}$$
$$150x_A + 150x_B + 60x_C + 100x_D + \tag{3c}$$
$$125x_E + 100x_F + 50x_G + 80x_H \leq 600 \tag{3d}$$
$$x_A, \ldots, x_H \in \{0, 1\} \tag{3e}$$

Model 4: portfolio selection

Statement *An insurance company must select the investments to perform out of a given set of possible assets (stocks, bonds, etc. . .).*

The total budget is 600 *Keuros, and the capital required by each asset is:*

Asset	A	B	C	D	E	F	G	H
Capital	150	150	60	100	125	100	50	80

Each asset refers to a given sector and nation, and has an expected return on investment (ROI):

Asset	A	B	C	D	E	F	G	H
Nation	Germany	Italy	USA	Italy	Italy	France	Italy	UK
Sector	autom.	autom.	ICT	ICT	real estate	real estate	short bond	long bond
ROI	11%	9%	13%	10%	8%	7%	3%	5%

The internal rules impose:

- *to select at most* 5 *investements (to avoid excessive fragmentation)*
- *to select at most* 3 *investments in Italy, at most* 3 *abroad*
  (for geographical diversification)
- *to select a bond if an ICT investement is selected (to limit risk)*

The company aims to maximize the expected ROI, respecting the rules

Model

- $x_j = 1$ if investment $j \in \{1, \ldots, 8\}$ is performed;
  $x_j = 0$ otherwise

$$\max f(x) = 0.11 \cdot 150x_A + 0.09 \cdot 150x_B + 0.13 \cdot 60x_C + 0.10 \cdot 100x_D + \tag{4a}$$

$$0.08 \cdot 125x_E + 0.07 \cdot 100x_F + 0.03 \cdot 50x_G + 0.05 \cdot 80x_H \tag{4b}$$

$$150x_A + 150x_B + 60x_C + 100x_D + \tag{4c}$$

$$125x_E + 100x_F + 50x_G + 80x_H \leq 600 \tag{4d}$$

$$x_A + x_B + x_C + x_D + x_E + x_F + x_G + x_H \leq 5 \tag{4e}$$

$$x_B + x_D + x_E + x_G \leq 3 \tag{4f}$$

$$x_A + x_C + x_F + x_H \leq 3 \tag{4g}$$

$$x_C \leq x_G + x_H \tag{4h}$$

$$x_D \leq x_G + x_H \tag{4i}$$

$$x_A, \ldots, x_H \in \{0, 1\} \tag{4j}$$

Model 5: assignment

Statement *The production process in a factory requires* 3 *parallel independent tasks.*

Each task requires exactly one machine, and each machine can perform at most one task at a time. To save time, the tasks are performed in parallel on different machines.

The cost required to perform each task on each machine is reported in the following table:

	M_1	M_2	M_3
T_1	2	6	3
T_2	8	4	9
T_3	5	7	8

The factory managers want to assign the tasks to the machines so as to minimize the total cost.

Model

- $x_{i,j} = 1$ if task $i \in \{1, 2, 3\}$ is assigned to machine $j \in \{1, 2, 3\}$; $x_{i,j} = 0$ otherwise

$$\min f(x) = 2x_{1,1} + 6x_{1,2} + 3x_{1,3} + \quad (5a)$$

$$8x_{2,1} + 4x_{2,2} + 9x_{2,3} + \quad (5b)$$

$$5x_{3,1} + 7x_{3,2} + 8x_{3,3} \quad (5c)$$

$$x_{1,1} + x_{2,1} + x_{3,1} = 1 \quad (5d)$$

$$x_{1,2} + x_{2,2} + x_{3,2} = 1 \quad (5e)$$

$$x_{1,3} + x_{2,3} + x_{3,3} = 1 \quad (5f)$$

$$x_{1,1} + x_{1,2} + x_{1,3} = 1 \quad (5g)$$

$$x_{2,1} + x_{2,2} + x_{2,3} = 1 \quad (5h)$$

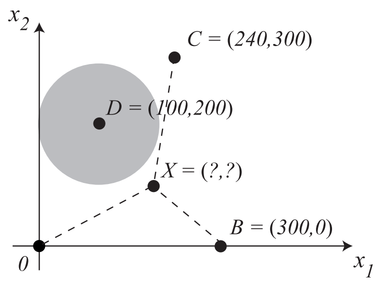
$$x_{3,1} + x_{3,2} + x_{3,3} = 1 \quad (5i)$$

$$x_{i,j} \in \{0, 1\} \qquad i, j \in \{1, 2, 3\} \quad (5j)$$

Model 6: facility location

Statement *Oil is extracted from three pits, located in positions* $A = (0, 0)$, $B = (300, 0)$ *and* $C = (240, 300)$.

A refinery must be built and connected to the three pits;
the refinery must be at least 100 *km away from position* $D = (100, 200)$;
the oil pipelines can be built everywhere.



Decide the location of the refinery, so as to minimize the total pipeline cost, which is proportional its length.

Model

- x_1 and x_2: cartesian coordinates of the refinery

$$\min f(x) = \left[(x_1 - 0)^2 + (x_2 - 0)^2\right] + \tag{6a}$$

$$\left[(x_1 - 300)^2 + (x_2 - 0)^2\right] + \tag{6b}$$

$$\left[(x_1 - 240)^2 + (x_2 - 300)^2\right] \tag{6c}$$

$$\sqrt{(x_1 - 100)^2 + (x_2 - 200)^2} \geq 100 \tag{6d}$$

Model 7: diet

Statement *You go to the market to buy food for today; you can buy bread, milk, eggs, meat and dessert.*

Bread costs 2 *euros/hg, milk* 3 *euros/hg, eggs* 4 *euros/hg, meat* 19 *euros/hg and dessert* 20 *euros/hg.*

You want to gain a minimum amount of nutrients from the food you buy: at least 2 000 *calories,* 50 *g of proteins and* 700 *mg of calcium.*

*The following table reports the amount of nutrients for each available food (*phantasy values, of course. . . *):*

	bread	milk	eggs	meat	dessert
calories/hg	110	160	180	260	420
proteins (g/hg)	4	8	13	14	4
calcium (mg/hg)	2	285	54	80	22

Determine the amounts of food to buy so as to minimize the total cost, while respecting the nutrition requirement.

Model

- x_1 = hg of bread bought
- x_2 = hg of milk bought
- x_3 = hg of eggs bought
- x_4 = hg of meat bought
- x_5 = hg of dessert bought

$$\min f(x) = 2x_1 + 3x_2 + 4x_3 + 19x_4 + 20x_5 \tag{7a}$$
$$110x_1 + 160x_2 + 180x_3 + 260x_4 + 420x_5 \geq 2\,000 \tag{7b}$$
$$4x_1 + 8x_2 + 13x_3 + 14x_4 + 4x_5 \geq 50 \tag{7c}$$
$$2x_1 + 285x_2 + 54x_3 + 80x_4 + 22x_5 \geq 700 \tag{7d}$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0 \tag{7e}$$

Model 8: multi-period production

Statement *A factory produces a single product, at a unitary production cost which changes month after month as follows:*

	Jan	Feb	Mar	Apr	May	Jun
Prod. cost (euros/unit)	20	25	30	40	50	60

The factory can produce a different maximum amount in each month:

	Jan	Feb	Mar	Apr	May	Jun
Prod. capacity (units)	1 500	2 000	2 200	3 000	2 700	2 500

The products are sold in blocks at the end of each month;
the demand and the price are known in advance:

	Jan	Feb	Mar	Apr	May	Jun
Demand (units)	1 100	1 500	1 800	1 600	2 300	2 500
Price (euros/unit)	180	180	250	270	300	320

The products are first stored in a temporary depot for free; at the end of the month, the unsold ones are moved to a larger depot which can store 3 000 *units at* 2 *euros/unit per month.*

Determine the production plan in order to maximize the total profit (total revenue minus total production and storage costs.

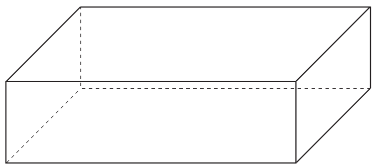
Model

- x_i = number of products for month $i \in \{1, \dots, 6\}$
- y_i = number of products stored at the beginning of month $i \in \{1, \dots, 7\}$ (i. e., after selling the products and moving the unsold ones to the depot)

$$
\begin{aligned}
\max f(x) = (180\ 1100 + 180\ 1500 + 250\ 1800 + & \quad (8a)\\
270\ 1600 + 300\ 2300 + 320\ 2500) - & \quad (8b)\\
-(20x_1 + 25x_2 + 30x_3 + 40x_4 + 50x_5 + 60x_6) + & \quad (8c)\\
-2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) & \quad (8d)\\
x_1 \le 1500 & \quad (8e)\\
x_2 \le 2000 & \quad (8f)\\
x_3 \le 2200 & \quad (8g)\\
x_4 \le 3000 & \quad (8h)\\
x_5 \le 2700 & \quad (8i)\\
x_6 \le 2500 & \quad (8j)\\
y_i \le 3000 \qquad i \in \{1, \ldots, 6\} & \quad (8k)\\
y_1 = y_7 = 0 & \quad (8l)\\
y_i + x_i - d_i = y_{i+1} \qquad i \in \{1, \ldots, 6\} & \quad (8m)\\
x_i \ge 0 \qquad i \in \{1, \ldots, 6\} & \quad (8n)\\
y_i \ge 0 \qquad i \in \{1, \ldots, 7\} & \quad (8o)
\end{aligned}
$$

Model 9: car design

Statement *You must design a new concept car, named* Spatium*, which will be shaped as a rectangular parallelepiped*



The cost of the car has already been decided; it implies that the total surface of the car must be lower than $37m^2$*.*

The length of the car must be at least $4m$*.*

The width of the car must be between $1.5m$ *and* $2.5m$*.*

The height of the car must be at most equal to the width and at most $1/3$ *of the length.*

You want to maximize the volume of the car, expressed in litres.

Model

- x_1: length of the car
- x_2: width of the car
- x_3: heigth of the car

$$\max f(x) = \frac{1}{1\,000} x_1 x_2 x_3 \tag{9a}$$

$$2(x_1x_2 + x_1x_3 + x_2x_3) \leq 37 \tag{9b}$$

$$x_1 \geq 4 \tag{9c}$$

$$x_2 \geq 1.5 \tag{9d}$$

$$x_2 \leq 2.5 \tag{9e}$$

$$x_3 \leq x_2 \tag{9f}$$

$$x_3 \leq \frac{1}{3} x_1 \tag{9g}$$

$$x_1, x_2, x_3 \geq 0 \tag{9h}$$