Foundations of Operations Research
Introduction to AMPL and Modellisation

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- Assignment Problem and Binary Variable
- Sensitivity Analysis

Assignment Problem

A company has 4 machines available for assignment to 4 tasks. All machines must be assigned one task, and each task requires processing by one machine. The time required to set up each machine for the processing of each task is given in the table below.

	TIME (hours)			
	Task1	Task2	Task3	Task4
Machine1	13	4	7	6
Machine2	1	11	5	4
Machine3	6	7	2	8
Machine4	1	3	5	9

The company wants to minimise the total setup time needed for the processing of all four tasks.

Set covering Problem

In the set covering problem, we are given a set $\mathbf{U}$, such that $|\mathbf{U}| = n$, and the sets $S_1, ..., S_k \subset \mathbf{U}$.

Set Cover: a set cover is a collection S of some of the sets from $\{S_i, i = 1..k\}$, whose union is the entire universe $\mathbf{U}$. Formally, S is a set cover if $\bigcup_{S_i \in S} S_i = \mathbf{U}$. If c_i is the cost associated to set S_i, we would like to select a set cover S with the minimum cost.

Data: $\mathbf{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $S_1 = \{1, 2, 3, 8, 9, 10\}$, $S_2 = \{3, 4, 5\}$, $S_3 = \{4, 5, 7\}$, $S_4 = \{5, 6, 7\}$, $S_5 = \{6, 7, 8, 9, 10\}$; $\{c_i, i = 1..5\} = \{16, 12, 18, 20, 10\}$.

Sensitivity Analysis

Sometimes we might want to know how to modify the problem parameter in order to obtain a certain type of solution. Alternatively we might want to **know what would be the effect of a parameter's modification on the solution**, and whether it is worthwhile relaxing some constraints.
There are two useful quantities we can extract from the result of an optimised solution:

- **Slack variable**: when a solver solves a linear problem it transforms **inequalities into equalities** through the use of a slack variable:

  $$ax \leq b \quad \Rightarrow \quad ax + s = b$$

  After solving the problem one can obtain the value of the slack variable for each inequality of the problem: those for which $s = 0$ are said to be **binding** or **active**, meaning it is exactly verified, e.g. $ax = b$. This is useful when one wants to know which constraints are actually limiting the search for the optimum.

Sensitivity Analysis

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There are two useful quantities we can extract from the result of an optimised solution:

- **Reduced cost**: usually the profit for some variables is such that the variable is set to 0 in the optimal solution. The reduced cost indicates the necessary improvement in the cost coefficient for said variable, in order for it to start having a non-0 value.
  For a maximisation problem it means how much one must increase the coefficient while for minimisation it means how much one must decrease it.

Sensitivity Analysis

Sometimes we might want to know how to modify the problem parameter in order to obtain a certain type of solution. Alternatively we might want to **know what would be the effect of a parameter's modification on the solution**, and whether it is worthwhile relaxing some constraints. There are two useful quantities we can extract from the result of an optimised solution:

- **Shadow price**: the less a problem is constrained, the better the value of the objective function. When we relax the value of some constraint parameter, we have a chance to improve the objective value: the shadow price is the change in the objective value one would obtain **per unit of constraint parameter** that we change. This is also called **dual variable**.
  Note that only **active constraints** will give a non 0 shadow price: if the constraint is not constraining already, there is no sense in changing its bound.

AMPL commands

Let us list some useful AMPL commands to analyse the solution of the problem at hand:

- `display var.rc;` : displays the reduced cost of the variable `var`.
- `display var.current;` : displays the current value of the objective function coefficient of variable `var`.
- `display ctr.slack;` : displays the value of the slack variable of constraint `ctr`.
- `display ctr;` : displays the shadow price (or dual variable) of constraint `ctr`.
- `display ctr.current;` : displays the current value of the right-hand side of constraint `ctr`.
- `reset;` : resets the LP model in memory.
- `reset data;` : resets the numerical values of the parameters, retaining the model.

Practical Example

A company has to determine the **best number of three models of a product to produce** in order to maximize profits. Constraints include **production capacity limitations** (time available in minutes) in each of three departments (cutting and dyeing, sewing, and inspection and packaging) as well as constraint that requires the production of at least 1000 units of first model. The linear programming model is shown here:

$$\text{Max} \quad 3x_1 + 5x_2 + 4.5x_3$$

$$12x_1 + 10x_2 + 8x_3 \leq 18000 \quad \text{Cutting and dyeing}$$

$$15x_1 + 15x_2 + 12x_3 \leq 18000 \quad \text{Sewing}$$

$$3x_1 + 4x_2 + 2x_3 \leq 9000 \quad \text{Inspection and modeling}$$

$$x_1 \geq 1000$$

$$x_1, x_2, x_3 \geq 0$$

Practical Example

- What is the value of the solution variables at the optimum ?
- How much would the profit contribution for model 2 have to change to make it worthwhile to produce some units?
- Which constraints are binding ? (which constraints have a 0 slack variable) What does it mean ? (interpret for each constraint)
- What are the shadow prices for the four constraints ? Interpret the value for the fourth constraint.
- Can you guess how we should modify the profit for model 1 in order to modify the solution ? Should we increase it by $1, what would be the increase in the objective value?
- Overtime rates in the sewing department are **$12 per hour**. Would you recommend that the company consider using overtime in that department?

Post-optimality

We now know which to parameter to modify to obtain specific results (or leave the solution untouched) through reduced costs and shadow prices, but not **how much we can vary the coefficients** maintaining the effects anticipated from said costs and prices. **Post-optimality** studies allow us to determine the range of the coefficients for which the anticipated effect remains. In other words, should we violate these ranges, the "structure" of the solution changes: 0 valued variables can become non 0 or vice versa; in any case we'd have to recompute reduced costs and shadow prices.

- `display var.down;/display var.up;` : displays, respectively, the lower and upper endpoints of the range of the objective-function coefficient of the variable `var` for which the current basis remains optimal.
- `display ctr.down;/display ctr.up;` : Displays, respectively, the lower and upper endpoints of the range of the right-hand-side of constraint `ctr` for which the current basis remains feasible, and hence optimal.

Post-optimality

Use the preceding problem and extract:

- the bounds for the objective function coefficients;
- the bounds for the right-hand side coefficients of the constraints;

such that the structure of the solution is not changed.
Modifiy the value of a constraint parameter or an objective coefficient such that the structure of the solution is changed (i.e. $x_3 = 0$ or $x_2 \neq 0$).

Relaxation of Integer Problems

Modify the preceding model as follows (using now **integer variables**):

$$\text{Max} \quad 3x_1 + 5x_2 + 4.5x_3$$

$$12x_1 + 10x_2 + 8x_3 \leq 18000 \quad \text{Cutting and dyeing}$$

$$15x_1 + 15x_2 + 11x_3 \leq 18000 \quad \text{Sewing}$$

$$3x_1 + 4x_2 + 2x_3 \leq 9000 \quad \text{Inspection and modeling}$$

$$x_1 \geq 1000$$

$$x_1,\, x_2,\, x_3 \in \mathbb{Z}^+$$

Solve the problem and write down the objective value and variables. Solve the problem by **relaxing the integrality constraint**, i.e. using continuous positive variables: $x_1,\, x_2\; x_3 \geq 0$. Compare the value of the objective and of the variables: usually the result of a relaxed problem is more favourable and the solution is not integral.