1) Minimum spanning tree

Given the weighed undirected graph whose cost function is reported in the following table (where a dash represent the absence of an edge), determine the minimum spanning tree with Kruskal's algorithm.

	1	2	3	4	5	6
1	-	9	15	13	11	-
2	9	-	4	-	5	-
3	15	4	-	-	-	2
4	13	-	-	-	-	12
5	11	5	-	-	-	3
6	-	-	2	12	3	-

Determine it with Prim's algorithm, starting from each vertex.

Solution

Kruskal's algorithm :

$c_{36} = 2$ (OK) $c_{56} = 3$ (OK) $c_{23} = 4$ (OK) $c_{25} = 5$ (NO)
$c_{21} = 9$ (OK) $c_{15} = 11$ (NO) $c_{46} = 12$ (NO) $c_{14} = 13$ (NO)

Checking $(1, 4)$ is unnecessary

Total cost: $f^* = 30$

Prim's algorithm (starting from vertex 1) :

$c_{12} = 9 \quad c_{23} = 4 \quad c_{36} = 2 \quad c_{56} = 3 \quad c_{46} = 12 \quad \Rightarrow f^* = 30$

Prim's algorithm (starting from vertex 2) :

$c_{23} = 4 \quad c_{36} = 2 \quad c_{56} = 3 \quad c_{12} = 9 \quad c_{46} = 12 \quad \Rightarrow f^* = 30$

Prim's algorithm (starting from vertex 3) :

$c_{36} = 2 \quad c_{56} = 3 \quad c_{23} = 4 \quad c_{12} = 9 \quad c_{46} = 12 \quad \Rightarrow f^* = 30$

Prim's algorithm (starting from vertex 4) :

$c_{46} = 12 \quad c_{36} = 2 \quad c_{56} = 3 \quad c_{23} = 4 \quad c_{12} = 9 \quad \Rightarrow f^* = 30$

Prim's algorithm (starting from vertex 5) :

$c_{56} = 3 \quad c_{36} = 2 \quad c_{23} = 4 \quad c_{12} = 9 \quad c_{46} = 12 \quad \Rightarrow f^* = 30$

Prim's algorithm (starting from vertex 6) :

$c_{36} = 2 \quad c_{56} = 3 \quad c_{23} = 4 \quad c_{12} = 9 \quad c_{46} = 12 \quad \Rightarrow f^* = 30$

2) Shortest path (nonnegative costs)

Given a weighted directed graph whose cost function is reported in the following table (where a dash represents the absence of an arc), determine the set of all shortest paths from node 2 to the other nodes.

	1	2	3	4	5	6
1	-	3	2	4	4	5
2	3	-	-	2	-	5
3	2	-	-	-	1	2
4	4	2	-	-	10	16
5	4	-	1	10	-	4
6	5	5	2	16	4	-

Solution

1. Set $d_2 = 0$ and mark node 2
2. From node 2 reach nodes 1 (update $d_1 = 3$), 4 (update $d_4 = 2$) and 6 (update $d_6 = 5$); mark node 4
3. From node 4 reach nodes 1, 2, 5 (update $d_5 = 12$) and 6; mark node 1
4. From node 1 reach nodes 2, 3 (update $d_3 = 5$), 4, 5 (update $d_5 = 7$), and 6; mark node 3
5. From node 3 reach nodes 1, 5 (update $d_5 = 6$), and 6; mark node 6
6. From node 6 reach nodes 1, 2, 3, 4, and 5; mark node 5
7. All nodes have been reached: terminate

Shortest path arborescence:

i	1	2	3	4	5	6
d_i	3	0	5	2	6	5
π_i	2	-	1	2	3	2

3) Shortest path (general costs)

Given a weighted directed graph whose cost function is reported in the following table (where a dash represents the absence of an arc), determine the set of all shortest paths from node B to the other nodes.

	A	B	C	D
A	-	8	-	-6
B	8	-	6	2
C	-	2	-	-
D	6	5	-	-

Solution

Floyd-Warshall's algorithm starts with the direct paths (arcs).

	A	B	C	D
A	-	8	-	-6
B	8	-	6	2
C	-	2	-	-
D	6	5	-	-

Then, for each node $k \in N$, it scans all pairs (i, j) evaluating whether the alternative path (i, k, j) is better than the current path (i, j).

$k = A$ Of course for $i = A$ and $j = A$ nothing changes; for the other $3 \cdot 3$ paths something might change:

TO BE CONTINUED[1]

[1]This exercise is very long (Floyd-Warshall's complexity is $O\left(n^3\right)$): in an exam, only one or two steps with respect to k will be required.

4) Project planning

Given a project composed of 6 activities with durations and precedences reported in the following table:

Activity	Duration	Precedences
A	8	-
B	9	-
C	10	A
D	11	A
E	9	B
F	11	C,D
G	8	D,E
H	8	F,G

draw the *activity-on-nodes* representation of the project.

Determine the overall duration of the project and its critical paths.

What is the slack of activities B and G?

Solution

The alphabetic order already provides a topological ordering of the nodes (but other ones could be possible, and would provide the same solution).

i	s	A	B	C	D	E	F	G	H	t
e_i	0	0	0	8	8	9	19	19	30	38
l_i	0	0	4	9	8	13	19	22	30	38
σ_i	0	0	4	1	0	4	0	3	0	0

There is only one critical path: (s, A, D, F, H, t).

5) Maximum flow

Solve the maximum flow problem from the source node s to the sink node t of the graph whose arcs have the capacities reported in the following table (a dash indicates the absence of an arc between the row and column nodes).

	s	1	2	3	4	t
s	-	15	-	-	5	-
1	-	-	3	13	-	-
2	-	-	-	7	-	8
3	-	-	-	-	-	10
4	-	4	9	10	-	-
t	-	-	-	-	-	-

Solution

There are many possible solution processes and optimal final solutions. One is the following:

1. Use augmenting path $(s, 1, 2, t)$ with $\delta = 3$
2. Use augmenting path $(s, 1, 3, t)$ with $\delta = 10$
3. Use augmenting path $(s, 4, 2, t)$ with $\delta = 5$

Now the sink t is no longer reachable from s; the reachable subset is $S = \{s, 1, 3\}$.

The current flow function is given by the following table:

(i,j)	$(s,1)$	$(s,4)$	$(1,2)$	$(1,3)$	$(2,3)$	$(2,t)$	$(3,t)$	$(4,1)$	$(4,2)$	$(4,3)$
x_{ij}	13	5	3	10	7	8	10	0	5	10

The flow value for S is $\phi_S = \phi_S^+ - \phi_S^- = 18 - 0$, where $\phi_S^+ = x_{s4} + x_{12} + x_{3t} = 5 + 3 + 10 = 18$ and $\phi_S^- = x_{41} + x_{23} = 0 + 0 = 0$.

The capacity of the cut induced by S is $k_S = k_{s4} + k_{12} + k_{3t} = 5 + 3 + 10 = 18$. Since flow and cut are identical, they are both optimal (maximum flow and minimum cut).

6) Standard form of LP

Given the following linear programming problem:

$$
\begin{aligned}
\max z = 2x_1 + x_2 \\
x_1 + x_2 \leq 4 \\
3x_1 - 2x_2 \geq 9 \\
2x_1 + x_2 = 16 \\
x_1 \geq 5 \\
x_2 \leq 0
\end{aligned}
$$

express the problem in standard form.

Solution

There are several ways to reduce the problem in standard form. In particular:

1. constraint $x_1 \geq 5$ can be seen as an additional constraint on a nonnegative variable x_1, or as a disguised nonnegativity constraint on an auxiliary variable $x_1' = x_1 - 5$;

2. constraint $x_2 \leq 0$ can be seen as an additional constraint on a free variable x_2, or as a disguised nonnegativity constraint on an auxiliary variable $x_2' = -x_2$;

3. constraint $2x_1 + x_2 = 16$ can be left as it is, or it can be used to remove a free variable so as to simplify the problem.

A smart reduction is to choose the second option in cases 1 and 2. In this case, we cannot choose the second option in case 3, since there is no free variable left.

$$
\begin{aligned}
\min z' = -2x_1' + x_2' - 10 \\
x_1' - x_2' + x_3 = -1 \\
3x_1' + 2x_2' - x_4 = -6 \\
2x_1' - x_2' = 6 \\
x_1', x_2', x_3, x_4 \geq 0
\end{aligned}
$$

An alternative smart reduction chooses the second option in cases 1 and 3, thus forcing the first option in case 2. In other words, x_1 would be replaced by $x' + 5$, but x_2 would be treated as a free variable, and replaced by the expression derived from constraint $2x_1 + x_2 = 16$ ($x_2 = 16 - 2x_1 = 6 - 2x_1'$). However, constraint $x_2 \leq 0$ should not be neglected.

7) Graphical solution of LP

Solve the following LP problem with the graphical method:

$$\begin{aligned} \max z = 2x_1 + 3x_2 & \\ 2x_1 + x_2 &\leq 5 \\ 2x_1 - 3x_2 &\leq 3 \\ -4x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

The graphical representation of the feasible region is the irregular pentangle with vertices $(0,0)$, $(3/2,0)$, $(9/4,1/2)$, $(1/2,4)$ and $(0,1)$.

The optimal solution is $x^* = (1/2, 4)$ with $f^* = 2x_1^* + 3x_2^* = 13$.

8) Simplex algorithm

1) Solve the following LP problem with the simplex algorithm:

$$\begin{aligned}\max z = 2x_1 - x_2 \\ 3x_1 - 2x_2 &\leq 0 \\ x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0\end{aligned}$$

2) Solve the following LP problem with the simplex algorithm:

$$\begin{aligned}\min z = x_1 - 2x_2 \\ -3x_1 + 3x_2 &\leq 1 \\ +x_1 - x_2 &\leq 5 \\ x_1, x_2 &\geq 0\end{aligned}$$

What can be deduced about the dual problem? Write the dual problem and solve it graphically to confirm the deduction.

Solution 1

0	-2	1	0	0
0	3	-2	1	0
8	1	2	0	1

Pivot on element $(1, 2)$: the basis changes, but the solution is the same (degenerate basic solution).

0	0	-1/3	2/3	0
0	1	-2/3	1/3	0
8	0	8/3	-1/3	1

Pivot on element $(2, 2)$: this time, both the basis and the solution change.

1	0	0	5/8	1/8
2	1	0	1/4	1/4
3	0	1	-1/8	3/8

The current solution is $(2, 3)$ and it is optimal (basic, feasible and with non-negative reduced costs). Its value is -1 ($+1$ for the original objective function).

Solution 2

0	1	-2	0	0
1	-3	3	1	0
5	1	-1	0	1

Pivot on element $(1, 2)$

2/3	-1	0	2/3	0
1/3	-1	1	1/3	0
16/3	0	0	1/3	1

The problem is unbounded (first column non positive). Correspondingly, the dual problem is unfeasible. In fact, the dual problem is

$$\begin{aligned} \max w = y_3 + 5y_4 & & \\ -3y_3 + y_4 &\leq 1 \\ +3y_3 - y_4 &\leq -2 \\ y_3, y_4 &\leq 0 \end{aligned}$$

and its two constraints are incompatible: $-3y_3+y_4 \leq 1$ is equivalent to $3y_3-y_4 \geq -1$, but the second constaint requires that $3y_3 - y_4 \leq -2$.

9) Duality

Write the dual of the following *LP* problem:

$$\begin{aligned}
\max z = 2x_1 + 3x_2 \\
2x_1 + x_2 \leq 5 \\
2x_1 - 3x_2 \leq 3 \\
-4x_1 + x_2 \leq 2 \\
x_1, x_2 \geq 0
\end{aligned}$$

Find a lower and an upper bound on the optimum of the problem.

Solution

$$\begin{aligned}
\min w = 5y_3 + 3y_4 + 2y_5 \\
2y_3 + 2y_4 - 4y_5 \geq 2 \\
y_3 - 3y_4 + y_5 \geq 3 \\
y_3, y_4, y_5 \geq 0
\end{aligned}$$

- Any feasible solution of the maximization problem provides a lower bound on the optimum; for example, $x = (0, 0) \Rightarrow z = 0 \leq z^*$
- Any feasible solution of the minimization problem provides an upper bound on the optimum; for example, $y = (3, 0, 0) \Rightarrow w = 15 \geq w^*$

10) Complementary slackness

Given the following *LP* problem:

$$\begin{aligned}\min z = x_1 + 4x_2 + 7x_3 + x_4& \\ -2x_1 + 2x_2 + x_3 + x_4 &\geq 2 \\ x_1 \qquad + x_3 \quad + x_4 &\geq 3 \\ x_1 - x_2 + x_3 - x_4 &\geq 1 \\ x_1, x_2, x_3 &\geq 0, x_4 \text{ free}\end{aligned}$$

write the dual problem and solve it graphically. Find the optimal primal solution through the complementary slackness conditions.

Solution

The given problem has an objective function to be minimized, constraints of the $\geq$ type, three nonnegative variables and a free variable (the fourth one). Consequently, the dual problem is a maximization problem, with nonnegative variables, three constraints of the $\leq$ type and an equality constraint (the fourth one).

$$\begin{aligned}\max w = 2y_1 + 3y_2 + 1y_3& \\ -2y_1 + y_2 + y_3 &\leq 1 \\ 2y_1 \qquad - y_3 &\leq 4 \\ y_1 + y_2 + y_3 &\leq 7 \\ y_1 + y_2 - y_3 &= 1 \\ y_1, y_2, y_3 &\geq 0\end{aligned}$$

Since one of the constraints is an equality, it is possible to compute one of its variables as a function of the other ones, *provided that the nonnegativity constraint is still taken into account* (the removed variable is not free). Thus, $y_1 + y_2 - y_3 = 1$ with $y_3 \geq 0$ yields $y_3 = y_1 + y_2 - 1 \geq 0$, that is

$$\begin{aligned}\max w = 3y_1 + 4y_2 - 1& \\ -y_1 + 2y_2 &\leq 2 \\ y_1 - y_2 &\leq 3 \\ 2y_1 + 2y_2 &\leq 8 \\ -y_1 - y_2 &\leq -1 \\ y_1, y_2 &\geq 0\end{aligned}$$

The choice of the variable to remove is arbitrary. A free variable is better, because it does not require to transform the nonnegativity constraint, but in the present case there was none.

The optimal solution of the dual problem is $A = (2, 2)$, while $E = (0, 1)$ is a degenerate basic solution because three variables assume zero value (y_1 and the two *slack* variables of constraints g_1 and g_4). Two separating hyperplanes are parallel, implying that one of the possible susets of $m = 4$ columns is not a basis. We could say that three of the basis solutions coincide and one of them is at infinity.

The complementary slackness conditions guarantee that, given two corresponding basic solutions of the primal and the dual problem, all products of corresponding variables in the two problem are zero. We use the original unsimplified formulation of the dual problem because the primal problem corresponds to it, and not to the simplified version.

$$\begin{array}{rcl}
x_1^* \left(1 + 2y_1^* - y_2^* - y_3^*\right) & = & 0 \\
x_2^* \left(4 - 2y_1^* + y_3^*\right) & = & 0 \\
x_3^* \left(7 - y_1^* - y_2^* - y_3^*\right) & = & 0 \\
x_4^* \left(1 - y_1^* - y_2^* + y_3^*\right) & = & 0 \\
y_1^* \left(-2x_1^* + 2x_2^* + x_3^* + x_4^* - 2\right) & = & 0 \\
y_2^* \left(x_1^* + x_3^* + x_4^* - 3\right) & = & 0 \\
y_3^* \left(x_1^* - x_2^* + x_3^* - x_4^* - 1\right) & = & 0
\end{array}$$

Since the optimal solution of the dual problem is $y^* = y_A = (2, 2, 3)$

$$\begin{array}{rcl}
x_1^* \left(1 + 4 - 2 - 3\right) & = & 0 \\
x_2^* \left(4 - 4 + 3\right) & = & 0 \\
x_3^* \left(7 - 2 - 2 - 3\right) & = & 0 \\
x_4^* \left(1 - 2 - 2 + 3\right) & = & 0 \\
2 \left(-2x_1^* + 2x_2^* + x_3^* + x_4^* - 2\right) & = & 0 \\
2 \left(x_1^* + x_3^* + x_4^* - 3\right) & = & 0 \\
3 \left(x_1^* - x_2^* + x_3^* - x_4^* - 1\right) & = & 0
\end{array}$$

from which

$$\begin{aligned}
0 &= 0\\
2x_2^* &= 0\\
0 &= 0\\
0 &= 0\\
-2x_1^* + 2x_2^* + x_3^* + x_4^* - 2 &= 0\\
x_1^* + x_3^* + x_4^* - 3 &= 0\\
x_1^* - x_2^* + x_3^* - x_4^* - 1 &= 0
\end{aligned}$$

which implies

$$\begin{aligned}
x_2^* &= 0\\
-2x_1^* + x_3^* + x_4^* - 2 &= 0\\
x_1^* + x_3^* + x_4^* - 3 &= 0\\
x_1^* + x_3^* - x_4^* - 1 &= 0
\end{aligned}$$

and (summing and subtracting the last two constraints)

$$\begin{aligned}
x_2^* &= 0\\
-2x_1^* + x_3^* &= 1\\
x_1^* + x_3^* &= 2\\
x_4^* &= 1
\end{aligned}$$

and finally $x^* = (1/3, 0, 5/3, 1)$.

The solution is certainly optimal for the primal, because it is feasible and the corresponding dual solution is also feasible and has the same value, which is optimal for the dual.

11) Gomory cuts

Given the following *ILP* problem:

$$\begin{aligned} \max z = x_1 - x_2 \\ 2x_1 - 3x_2 \leq 0 \\ 2x_1 + 3x_2 \leq 6 \\ x_1, x_2 \geq 0 \text{ e interi} \end{aligned}$$

and the optimal *tableau* of its continuous relaxation:

1/2	0	0	5/12	1/12
3/2	1	0	1/4	1/4
1	0	1	-1/6	1/6

generate a Gomory cut both in integer and fractionary form.

Add the cut to the *tableau* and reoptimize the problem.

Solution

1/2	0	0	5/12	1/12
3/2	1	0	1/4	1/4
1	0	1	-1/6	1/6

The optimal relaxed solution is $x^* = (3/2, 1)$ and its value is $z^* = 1/2$.

The Gomory cut derived from row 1 is $1/4x_3+1/4x_4 \geq 1/2$ (which corresponds to $x_1 \leq 1$ in the graphical representation).

1/2	0	0	5/12	1/12	1
3/2	1	0	1/4	1/4	0
1	0	1	-1/6	1/6	0
-1/2	0	0	-1/4	-1/4	1

The *pivot* element is $a_{34} = -1/4$ and the modified *tableau* becomes

1/3	0	0	1/3	0	1/3
1	1	0	0	0	1
2/3	0	1	-1/3	0	2/3
2	0	0	1	1	-4

The optimal relaxed solution is $x^* = (1, 2/3)$ and its value is $z^* = 1/3$.

The Gomory cut derived from row 1 is $2/3x_3+2/3x_5 \geq 2/3$ (which corresponds to $x_1 \leq x_2$ in the graphical representation).

1/3	0	0	1/3	0	1/3	0
1	1	0	0	0	1	0
2/3	0	1	-1/3	0	2/3	0
2	0	0	1	1	-4	0
-2/3	0	0	-2/3	0	-2/3	1

The *pivot* element can be either $a_{43} = -2/3$ or $a_{45} = -2/3$ indifferently (same ratio); we choose the former.

0	0	0	0	0	0	1/2
1	1	0	0	0	1	0
1	0	1	0	0	1	-1/2
1	0	0	0	1	-5	3/2
1	0	0	1	0	1	-3/2

The optimal relaxed solution is $x^* = (1, 1)$ and its value is $z^* = 0$. Since it is integer, it is optimal also for the original problem. Gomory's cutting plane algorithm terminates.

Note: To better understand the solution process, it is adviceable to draw the graphical representation of the problem, the basic solutions visited and the Gomory cuts generated. This is not required to solve the problem, but it helps understanding.

12) Branch-and-bound

Given the following *ILP* problem:

$$\begin{aligned}\max z = x_1 - x_2 \\ 2x_1 - 3x_2 \leq 0 \\ 2x_1 + 3x_2 \leq 6 \\ x_1, x_2 \geq 0 \text{ e interi}\end{aligned}$$

determine its optimal solution with the branch-and-bound method, solving graphically the continuous relaxation of its various subproblems. Draw the corresponding branching tree.

Solution

First notice that the problem is a maximization problem. This means that, with respect to the lesson slides, the superoptimal value is an upper bound instead of a lower bound. Conversely, the value of the heuristic solutions provide lower bounds, instead of upper ones. To avoid any confusion, it is possible to change the sign of the objective function at the beginning, solve the problem and go back to the original function at the end of the process. We will, on the contrary, keep the given function and adapt the terminology to the situation.

The continuous relaxation of the original problem P_0 has optimal solution $(3/2, 1)$, yielding an upper bound on the optimum equal to $UB_0 = 3/2 - 1 = 1/2$ (they can be found graphically).

Branching on x_1 produces subproblems:

1. P_1: $x_1 \leq \lfloor 3/2 \rfloor = 1$, whose continuous relaxation has optimal solution $(1, 2/3)$ yielding an upper bound on the optimum: $UB_1 = 1 - 2/3 = 1/3$;

2. P_2: $x_1 \leq \lceil 3/2 \rceil = 2$, whose continuous relaxation is unfeasible.

The only open branching node is P_1, which is further split into:

1. P_3: $x_2 \leq \lfloor 2/3 \rfloor = 0$, whose continuous relaxation has a single (optimal) solution $(0, 0)$ yielding an upper bound on the optimum: $UB_3 = 0 - 0 = 0$; this is also a lower bound, because both variables are integer: $LB = 0$;

2. P_4: $x_2 \leq \lceil 2/3 \rceil = 1$, whose continuous relaxation has optimal solution $(1, 1)$, yielding an upper bound on the optimum: $UB_4 = 0 - 0 = 0$; this is also a lower bound, because both variables are integer, but it does not improve the current lower bound (it is equal); anyway, the upper bound is not larger than the current lower bound, so the branching node can be closed.

No other branching node is open: the best known solution is optimal. This is (0.0) and its value is $LB = 0$. Another optimal solution is $(1, 1)$, but it was found later and the algorithm will return the first one.

13) Modelling and interpretation

A large bakery must define the production levels for its four main products: Biscuits, Fruitcakes, Plumcakes and Cakes. The selling prices of each product have already been fixed, respectively, to 2.5 Euros, 4 Euros, 4.3 Euros and 4.5 Euros, while the maximum demand is estimated to be 4 000 units for Biscuits, 2 000 units for Fruitcakes, 1 000 units for Plumcakes and 4 000 units for Cakes. The bakery employs five main ingredients: Flour, Milk, Jam, Eggs and Sugar. The composition of the products is reported in the following table:

	Biscuits	Fruitcakes	Plumcakes	Cakes
Flour	70%	40%	30%	55%
Milk	10%	10%	15%	20%
Jam	0%	40%	20%	0%
Egg	5%	0%	20%	10%
Sugar	15%	10%	15%	15%

The bakery has a limited availability of each ingredient: 5 000 units of Flour, 2 000 units of Milk, 2 500 units of Jam, 500 units of Eggs and 3 000 units of Sugar.

Provide a mathematical programming formulation for the problem of maximizing the income of the bakery.

Write the model in AMPL.

Given the AMPL output attached, answer the following questions:

a. how much Flour is still available in the end?

b. what's the value of the *slack* variable for the demand constraint on Plumcakes?

c. how much should the selling price of the Plumcakes increase in order to make their production profitable?

d. if 5 more Eggs were available, how much would the bakery income change?

e. is it profitable to invest in advertising to increase the demand of Biscuits?

f. if the demand estimate for Cakes were decreased to 2 000 units, should the production plan change?

Output

```
Level [*] :=
  Biscuits    3411.76
  Cakes       3294.12
  Fruitcakes  2000
  Plumcakes      0
;

Level.rc [*] :=
  Biscuits     0
  Cakes        0
  Fruitcakes   0
  Plumcakes   -4.22941
;

AvailabilityConstraints.slack [*] :=
  Eggs      0
  Flour     0
  Jam    1700
  Milk    800
  Sugar  1794.12
;

AvailabilityConstraints.dual [*] :=
  Eggs   41.7647
  Flour   0.588235
  Jam     0
  Milk    0
  Sugar   0
;

DemandConstraints.slack [*] :=
  Biscuits     588.235
  Cakes        705.882
  Fruitcakes     0
  Plumcakes   1000
;

DemandConstraints.dual [*] :=
  Biscuits    0
  Cakes       0
  Fruitcakes  3.76471
  Plumcakes   0
;
```

Solution (AMPL model)

Model file

```
set Ingredients;
```

```
set Products;

param Price{Products};

param Demand{Products};

param Availability{Ingredients};

param Composition{Ingredients,Products};

var Level{Products} >= 0;

maximize Income : sum{p in Products} Price[p] * Level[p];

subject to AvailabilityConstraints {i in Ingredients} :
  sum{p in Products} Composition[i,p] * Level[p] <= Availability[i];

subject to DemandConstraints {p in Products} :
  Level[p] <= Demand[p];

end;
```

Data file

```
set Ingredients := Flour Milk Jam Eggs Sugar;

set Products := Biscuits Fruitcakes Plumcakes Cakes;

param Price :=
  Biscuits    2.5
  Fruitcakes  4.0
  Plumcakes   4.3
  Cakes       4.5
;

param Demand :=
  Biscuits    4000
  Fruitcakes  2000
  Plumcakes   1000
  Cakes       4000
;

param Availability :=
  Flour  5000
  Milk   2000
  Jam    2500
  Eggs    500
  Sugar  3000
;
```

```
param Composition :
        Biscuits Fruitcakes Plumcakes Cakes :=
  Flour   0.70      0.40       0.30      0.55
  Milk    0.10      0.10       0.15      0.20
  Jam     0.00      0.40       0.20      0.00
  Eggs    0.05      0.00       0.20      0.10
  Sugar   0.15      0.10       0.15      0.15
;

end;
```

Solution (output interpretation)

a. how much Flour is still available in the end?
The remaining amount of Flour is given by the slack of the corresponding availability constraints `AvailabilityConstraints[Flour].slack`: it is zero. See also the corresponding shadow price (dual variable) `AvailabilityConstraints[Flour].dual`: it is strictly positive, which implies that the slack is zero by complementary slackness.

b. what's the value of the *slack* variable for the demand constraint on Plumcakes?
The slack variable is given by `DemandConstraints[Flour].slack`: it equals the total demand (1 000) because no Plumcakes are produced.

c. how much should the selling price of the Plumcakes increase in order to make their production profitable?
Plumcakes are currently not produced. Increasing the production of Plumcakes from 0 to ϵ would affect the income by -4.22941ϵ, because the reduced cost of Plumcakes is `Level[Plumcakes].rc`, that is -4.22941 (it is negative, instead of positive as we usually assume, because the objective is maximized, instead of minimized). In order to make the production profitable, the price must cover the additional cost increasing by 4.22941 Euros.

d. if 5 more Eggs were available, how much would the bakery income change?
A small increase in the availability of Eggs (an increase of 5 amounts to 1% of the current value, which is probably small) affects the income proportionally to the shadow price of the corresponding availability constraint `AvailabilityConstraints[Eggs].dual`, increasing it by $41.7647\epsilon = 208.8235$. If the additional Eggs can be found at price not larger than that, it is profitable to buy them, and increase the production. An increase is sufficiently small if it does not modify the optimal basis. Otherwise, the variation of the income is not proportional to the increase in availability, but smaller.

e. is it profitable to invest in advertising to increase the demand of Biscuits?
A small increase in the demand of Biscuits affects the income proportionally to the shadow price of the Biscuits demand constraint `DemandConstraints[Biscuits].dual`, that is 0. In fact the production of biscuits is lower than the maximum demand, there is a positive slack (corresponding to the zero shadow price by complementary slackness). Thus, an increase in the demand would not affect the production level: the advertising campaign would bring no advantage. If the shadow price were positive, it would provide a benchmark to estimate the profitability of the advertising campaign, given the cost and the effectiveness (impact on the demand) of the latter.

f. if the demand estimate for Cakes were decreased to 2 000 units, should the production plan change?
Presently, 3 294.12 Cakes are produced, and the maximum demand is 4 000; if the demand decreases by 2 000, it certainly affects the current production. It is not possible to say how much from the given information. If the decrease is at most $4\,000 - 3\,294.12 = 705.88$, on the contrary, the optimal production is not affected.