An Introduction to Tree Search

Acting rationally

- Introducing the **agent** and her **rationality**, two central concepts

- We need to build a system that can act rationally, where do we start?

AI approaches: the main ingredients

Problem

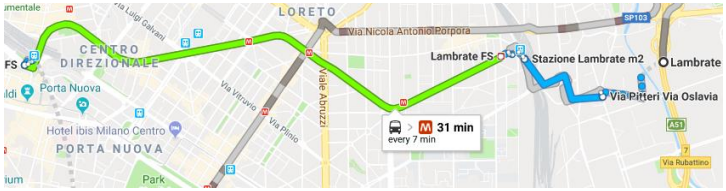
Solution

- Problems in AI have different dimensions of complexity
  - **computation**: some problems are difficult in the sense that designing an efficient algorithm for their resolution might be difficult or even not possible
  - **information**: the resolution of some problems might require the availability and the capability to process large amounts of data

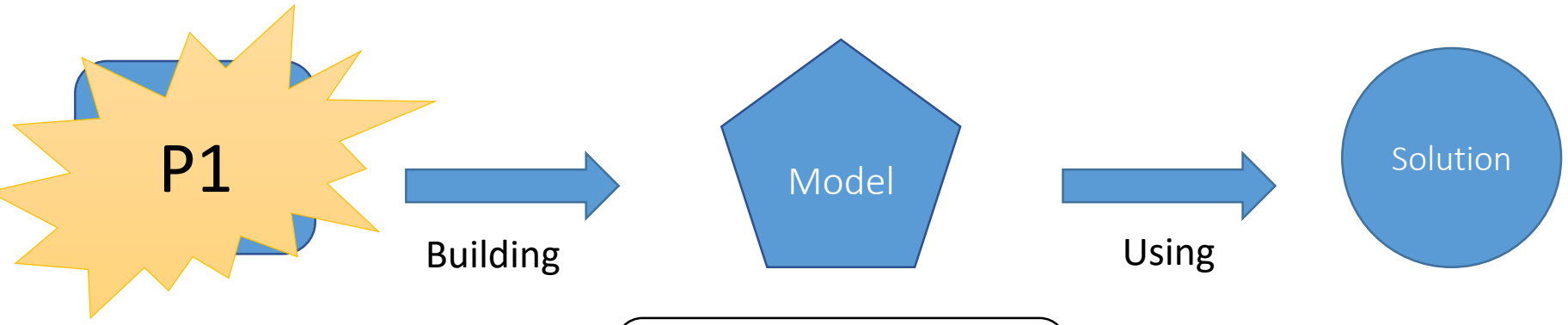
To have a more concrete view let's consider two (classical) running problems:

P1) Is there any person in this picture?

P2) What is the fastest route from A to B?



First approach: learning



- Building the model from data, selecting from a family
- Extracting relevant features and understanding how they map to a solution

how some features of the world map to the solution

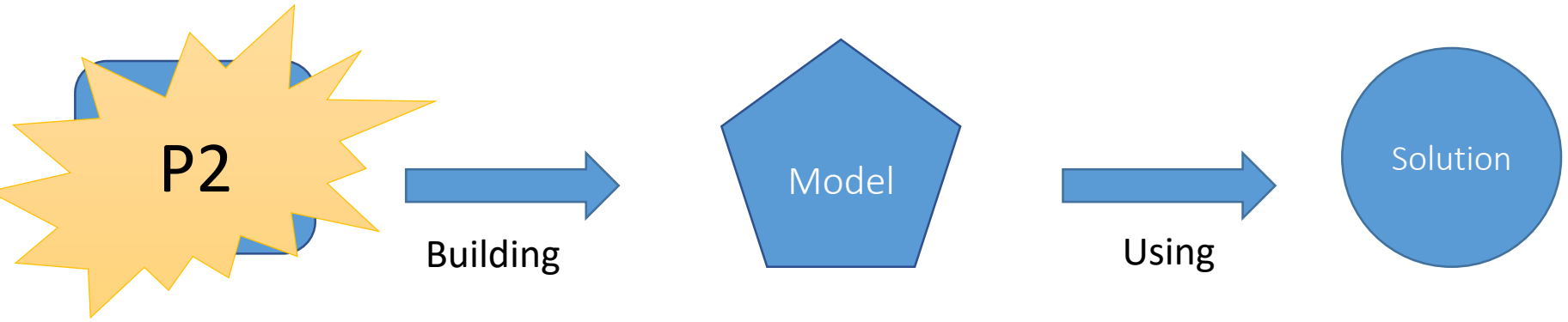
Agent: the entity that solves the problem

- The agent queries the model, she's a reactive or reflex agent

Typically difficult

Typically easy

Second approach: inference



- Building a model of the problem

descriptions of some features of the world and how they change

Agent: the entity that solves the problem

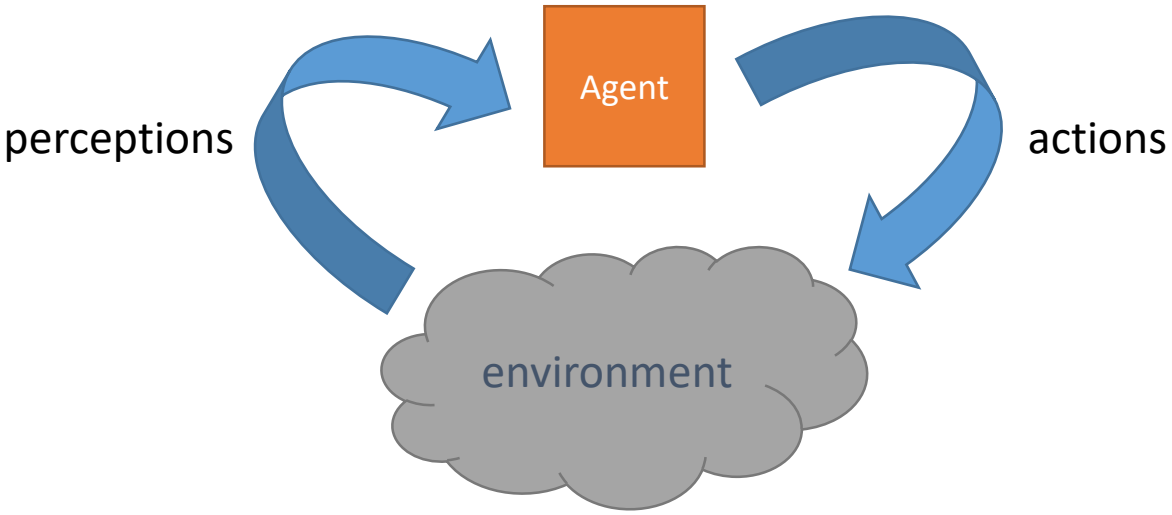
- Computing the solution by inference on the model
- The agent must search, reason, explore different directions

Typically easy

Typically difficult

- We will focus on this approach

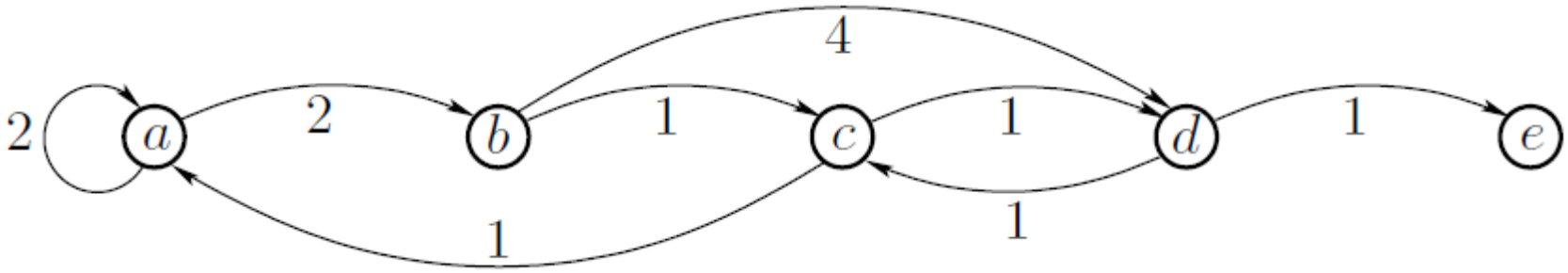
Agents



- The model we want to build is called **agent**
- The agent works on a problem model maintaining an internal representation of it integrating the environment perceptions
- It can perform actions that change the environment and, as a consequence, the internal representation
- The agent wants to accomplish something, it has goals/preferences and acts rationally with respect to them

State-based problem formulation

- (Single agent: the automated problem solver)
- State space defined as a set of **nodes**, each node represents a state; we assume a finite state space
- For each state, we have set of actions that can be undertaken by the agent from that state
- Transition model: given a starting state and an action, indicates an arrival state (here we assume no uncertainties, i.e., deterministic transitions and full observability)
- Action costs: any transition has a cost, which we assume to be greater than a positive constant (reasonable assumption, useful for deriving some properties of the algorithms we discuss)
- Initial state



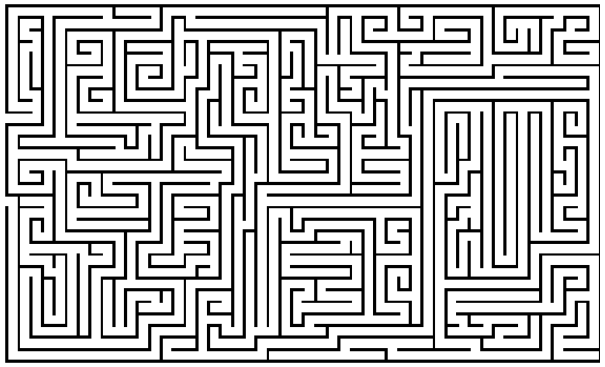
Compact representation: state transition graph G=(V,E)
(We will use "state" and "node" as interchangeable terms)

Formally describing the desired solution

- In the problem formulation we need to formally describe the features of the solution we seek

- Two (three) classes of problems:

feasibility

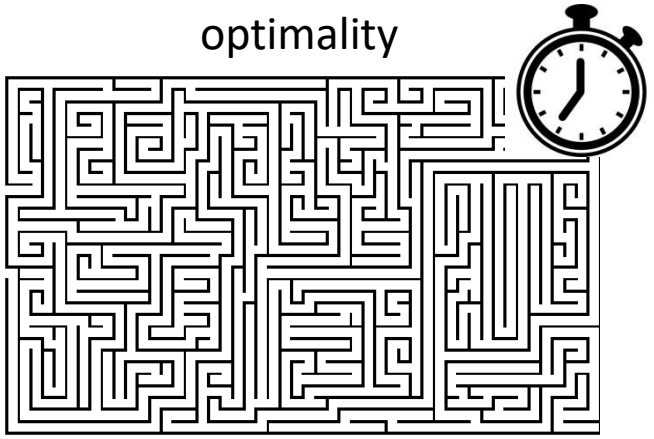


is there a path to an exit?

Set of goal states, find any sequence of actions (path) from the initial state to a goal state

(approximation)

optimality



If at least a path to an exit exists, what is the one with the minimum number of turns?

Set of goal states, find the sequence of actions (path) from the initial state to a goal state that has the minimum cost

Problem example

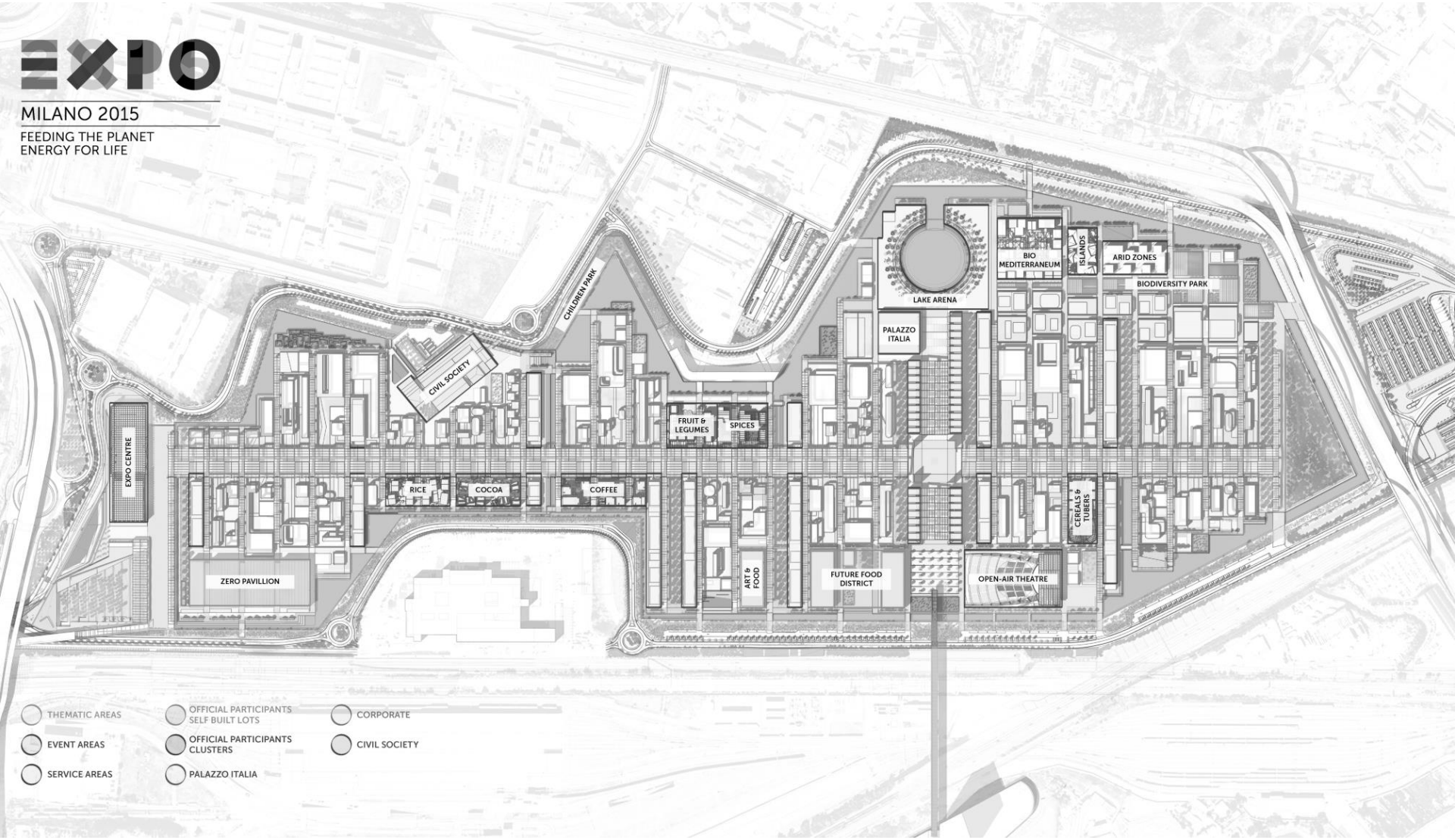
Consider a mobile robot moving on a graph-represented environment:

- **States**: nodes of the graph, they represent physical locations
- **Edges**: represent connections between nearby locations or, equivalently, movement actions
- **Initial state**: some starting location for the robot

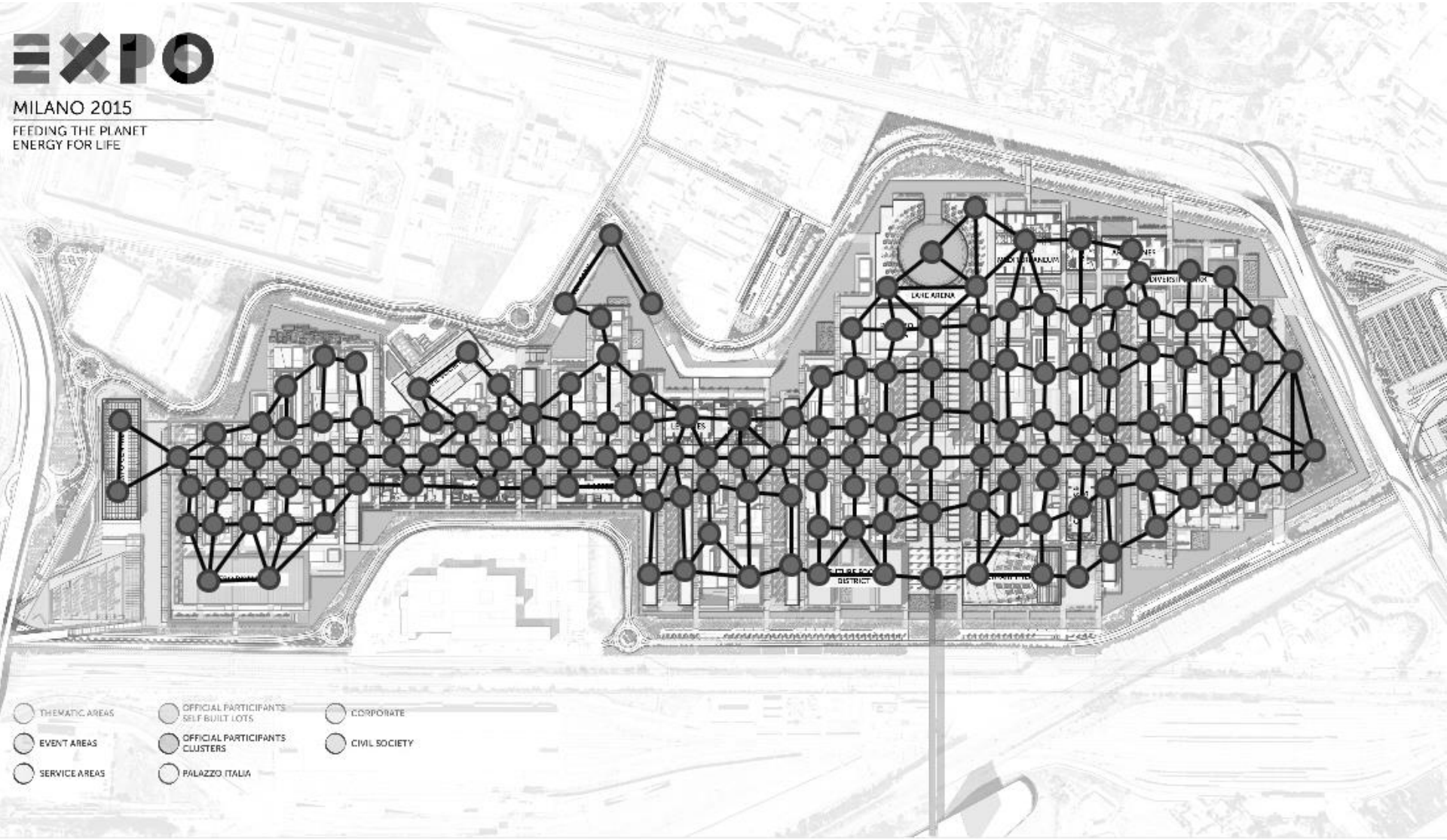
Desired solution:

- **Goal state(s)**: some location(s) to reach, e.g., recharging station, parking depot…
- Find a path to the initial location to a goal one

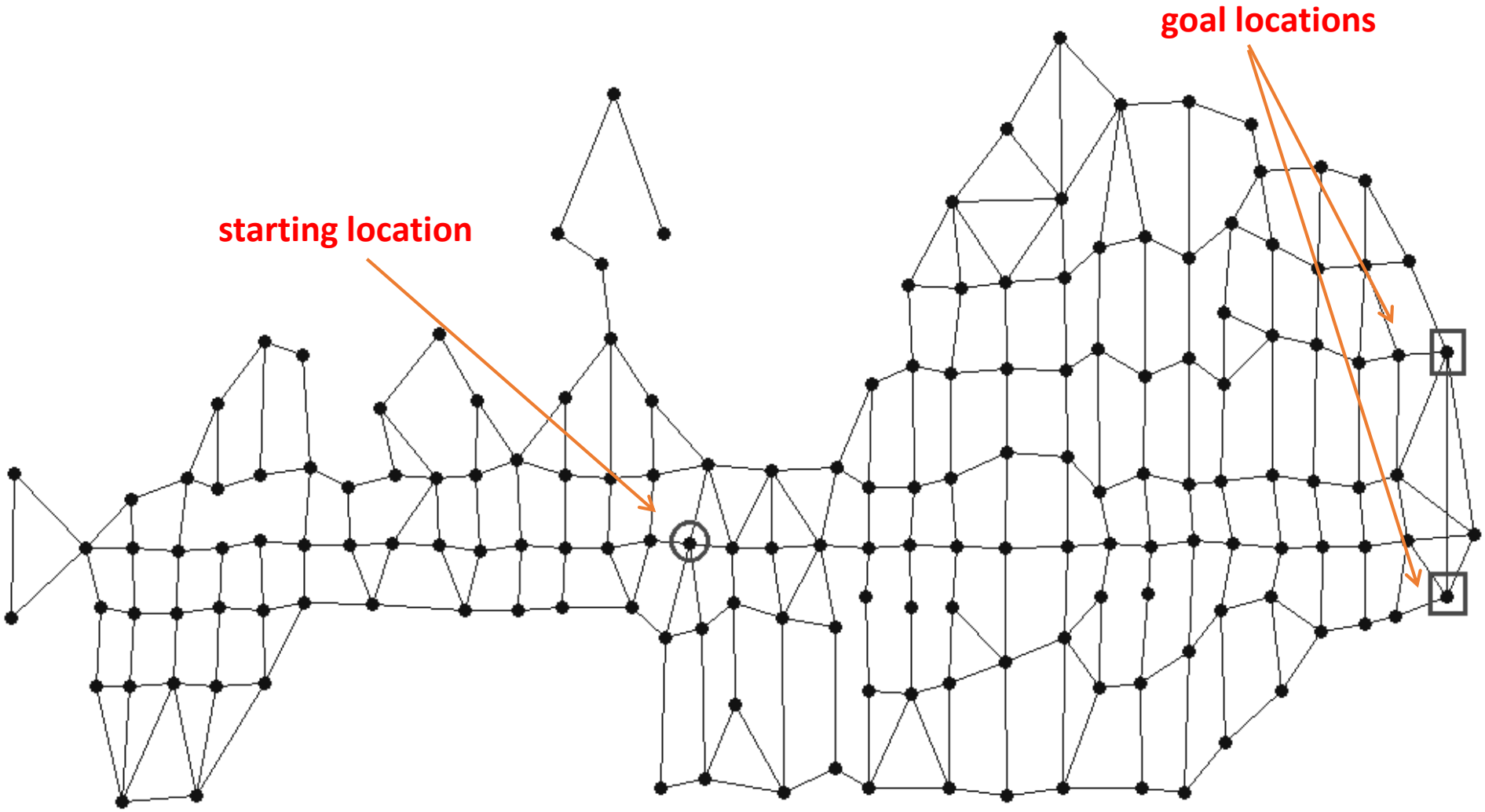
Problem example



Problem example

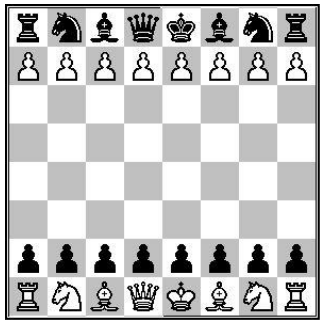


Problem example



Problem specification

- How to **specify** a planning problem?
- First approach: provide the full state transition graph G (as in the previous example)
- Most of the times this is not an affordable option due to the combinatorial nature of the state space:

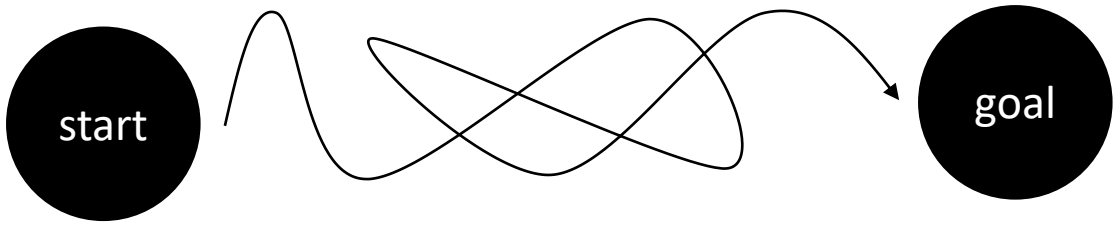


  - **Chess board**: approx. 10^{47} states (game tree approx. 10^{123})
  - We can specify the initial state and the transition function in some compact form (e.g., set of rules to generate next states)
  - The planning problem "unfolds" as search progresses
- We need an efficient procedure for *goal checking*

General features of search algorithms

A search algorithm explores the state-transition graph graph G until it discovers the desired solution

- In feasibility: when a goal node is visited the path that led to that node is returned
- In optimality: when a goal node is visited, if any other possible path to that node has higher cost the path that led to that node is returned



It does not suffice to visit a goal node, the algorithm has to reconstruct the path it followed to get there: it must keep a trace of its search

Such a trace can be mapped to a subgraph of G, it is called *search graph*

how to evaluate a (search) algorithm?

- We can evaluate a search algorithm along different dimensions
  - Sound?
  - Complete? (Systematic?)
  - Space complexity?
  - Time complexity?

(The above criteria can actually be used to evaluate a broader class of algorithms)

Soundness

- If the algorithm returns a solution, is it compliant with the desired features specified in the problem formulation?

- Example:
  - In feasibility: *does the returned solution lead to a goal?*
  - In optimality: *does the returned solution lead to a goal with minimum cost?*

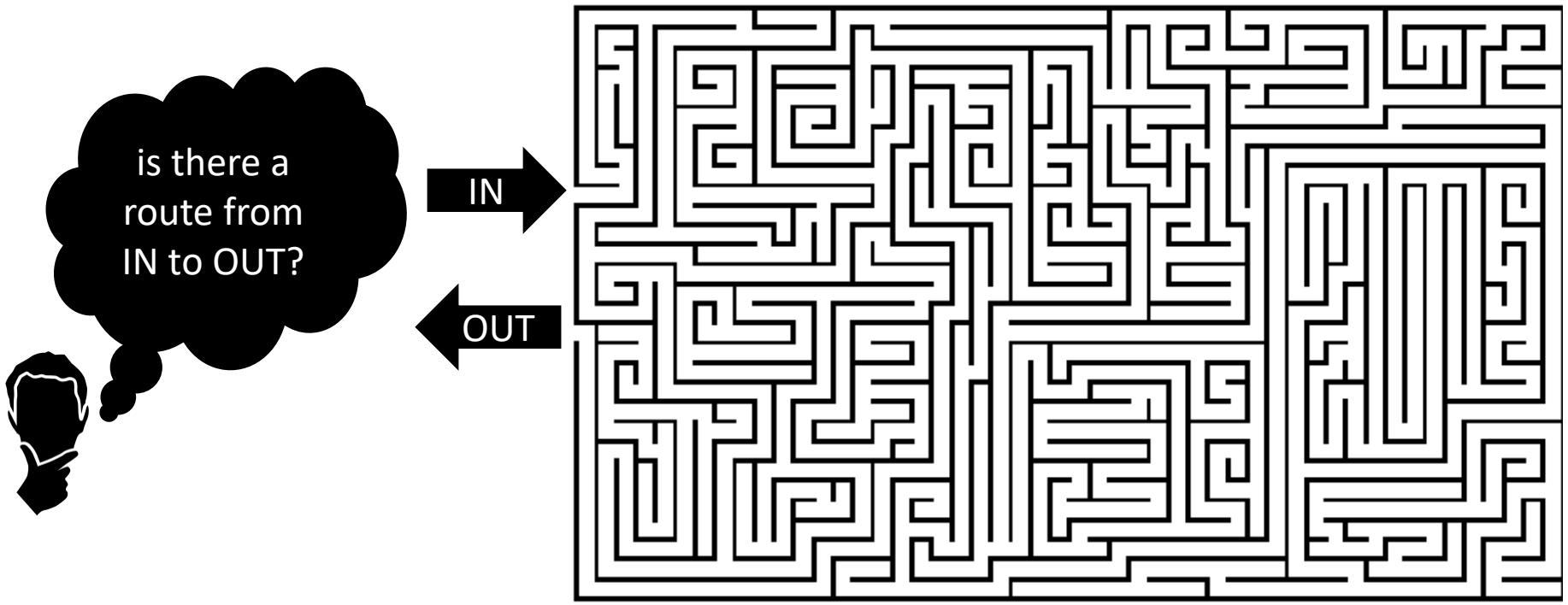
Completeness and the systematic property

- If a solution exists, does the algorithm find it?
- Example:
  - In feasibility: does it always find a path to the goal when it exists?
  - In optimality: does it always find the path to the goal that has minimum cost when at least one exists?

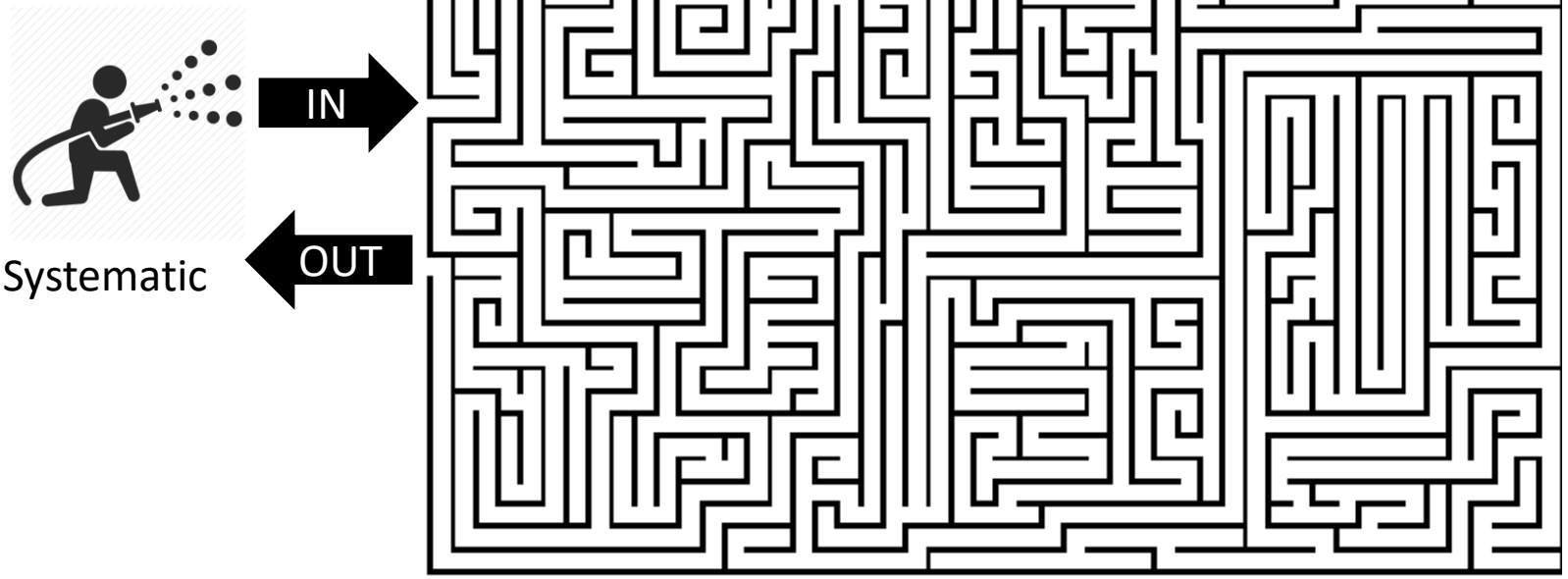
- Typically shown by proving that the search will/will not visit all states if given enough time

- If the state space is infinite, we can ask if the search is systematic:
  - if the answer is “yes” the algorithm must terminate
  - if the answer is “no”, it’s ok if it does not terminate but …
  - … all reachable states must be visited in the limit: as time goes to infinity, all states are visited (this definition is sound under the assumption of countable state space)

Visual example



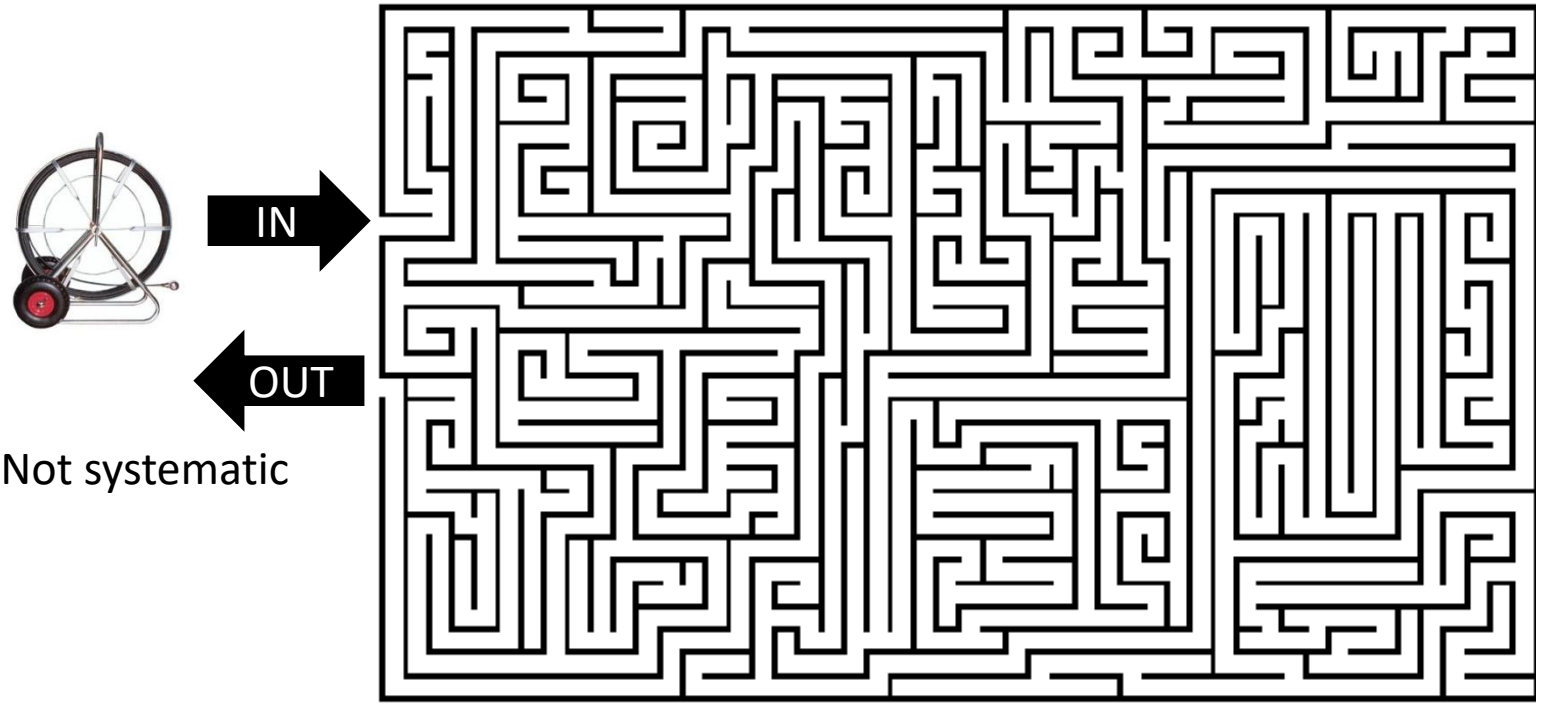
Visual example



Complete / Systematic

- Searching along **multiple** trajectories (either concurrently or not), eventually covers all the reachable space

Visual example

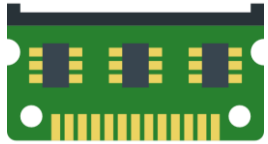


Not complete / Not systematic

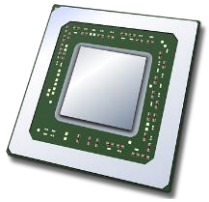
- Searching along a **single** trajectory, eventually gets stuck in a dead end

Space and time complexity

- Space complexity: how does the amount of memory required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Time complexity: how does the time required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Asymptotic trend:
  - We measure complexity with a function $f(n)$ of the input size
  - For analysis purposes, the "Big O" notation is convenient:

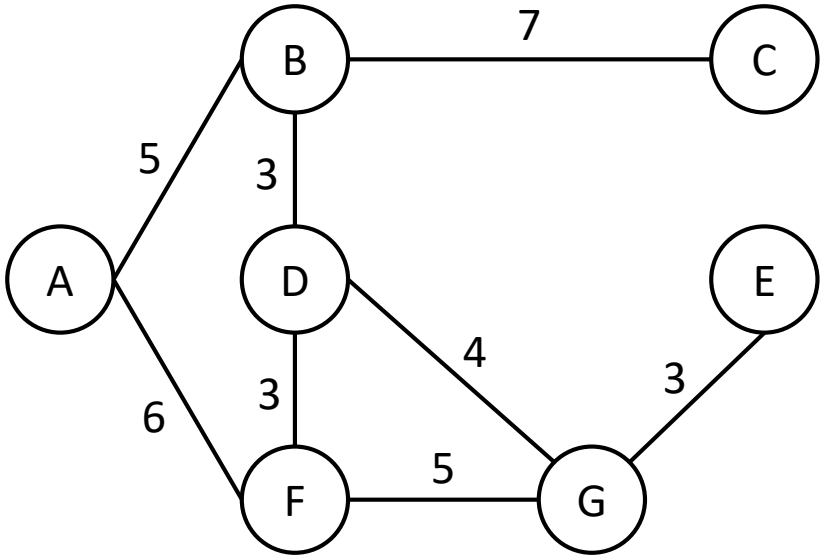
$$\text{A function } f(n) \text{ is } O(g(n)) \text{ if } \exists k > 0, n_0 \text{ such that } f(n) \leq kg(n) \text{ for } n > n_0$$

- An algorithm that is $O(n^2)$ is better than one that is $O(n^5)$
- If $g(n)$ is an exponential, the algorithm is not efficient

Running example

- To present the various search algorithms, we will use this *problem instance* as our running example

State-transition graph:



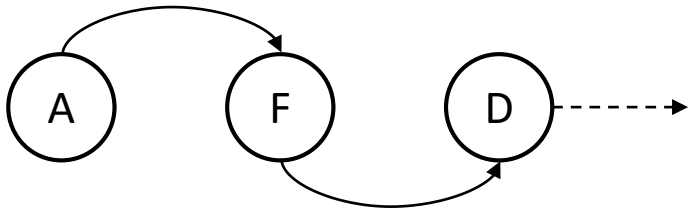
Initial state: A

Desired solution: any path to goal state E

- It might be useful to think it as a map, but keep in mind that this interpretation does not hold for every instance

Search algorithm definition

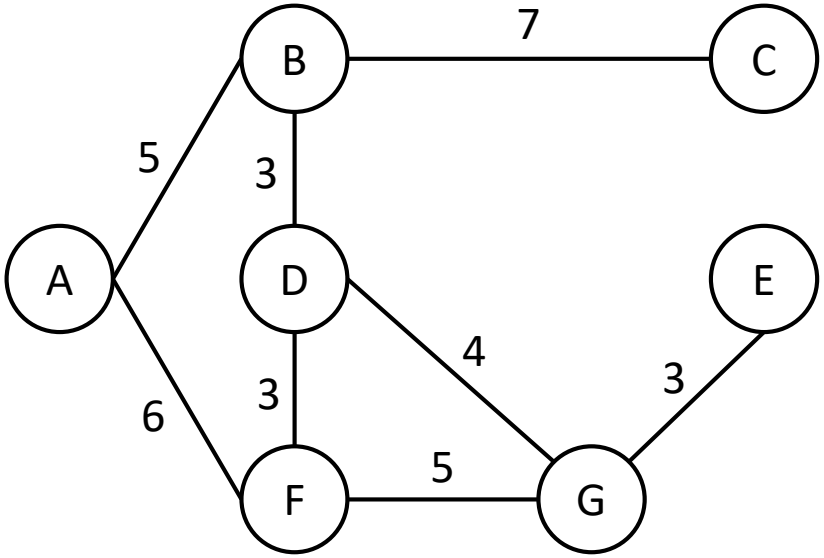
- The different search algorithms are substantially characterized by the answer they provide to the following question:

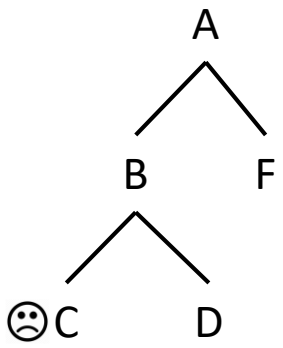


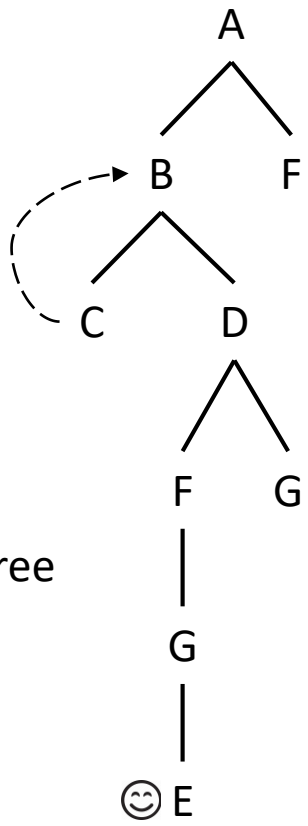
Given what I searched so far,
where to search next?

- The answer is encoded in a set of rules that drives the search and define its type, let's start with the simplest one

Depth-First Search (DFS)







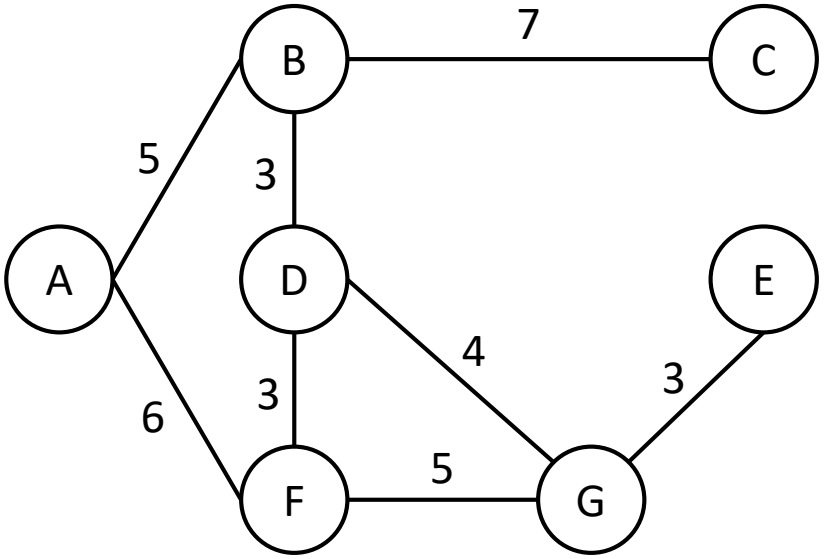
- A Depth-First Search (DFS) chooses the deepest node in the search tree (How to break ties? For now lexicographic order)
- We are **avoiding loops** on the same branch (loops are redundant paths)
- A dead end stopped the search, DFS seems not complete. Can we fix this?
- Let's endow our DFS with **backtracking**: a way to reconsider previously evaluated decisions

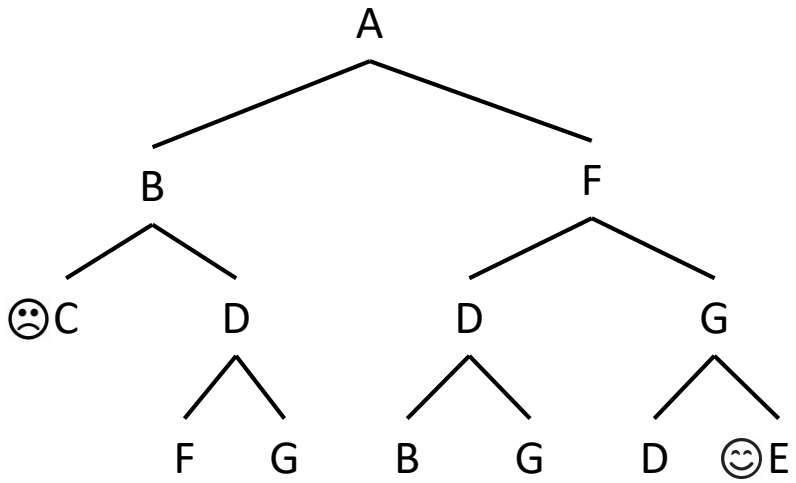
Solution: (A->B->D->F->G->E)

Depth-First Search (DFS)

- DFS with loops removal and BT is sound and complete
- Call b the maximum branching factor, i.e., the maximum number of actions available in a state
- Call d the maximum depth of a solution, i.e., the maximum number of actions in a path
- Space complexity: $O(d)$
- Time complexity: $1 + b + b^2 + \ldots + b^d = O(b^d)$

Breadth-First Search (BFS)



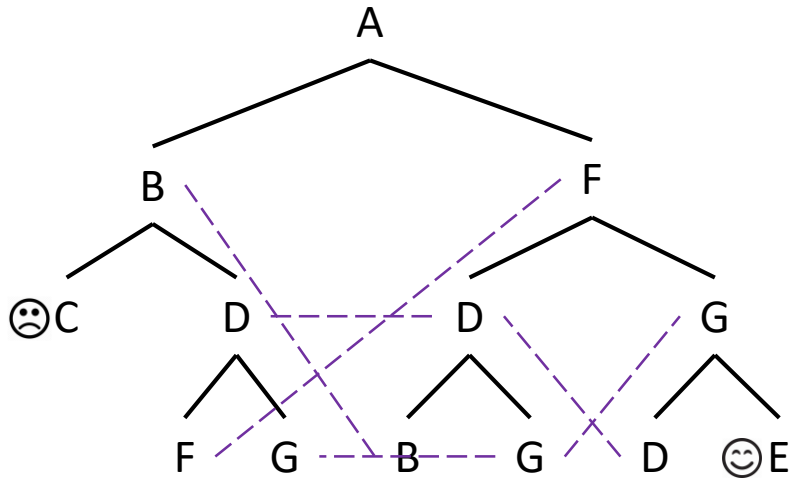


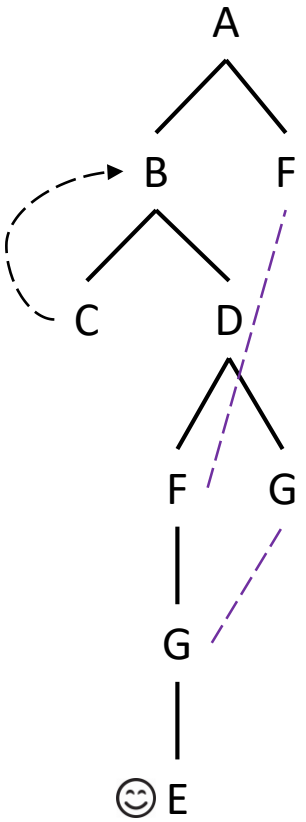
Solution: (A->F->G->E)

- A Breadth-First Search (BFS) chooses the shallowest node, thus exploring in a level by level fashion
- It has a more conservative behavior and does not need to reconsider decisions
- Call q the depth of the shallowest solution (in general $q \leq d$)
- Space complexity: $O(b^q)$
- Time complexity: $O(b^q)$

Redundant paths

- Both DFS and BFS visited some nodes multiple times (avoiding loops prevents this to happen only within the same branch)
- In general, this does not seem very efficient. Why?



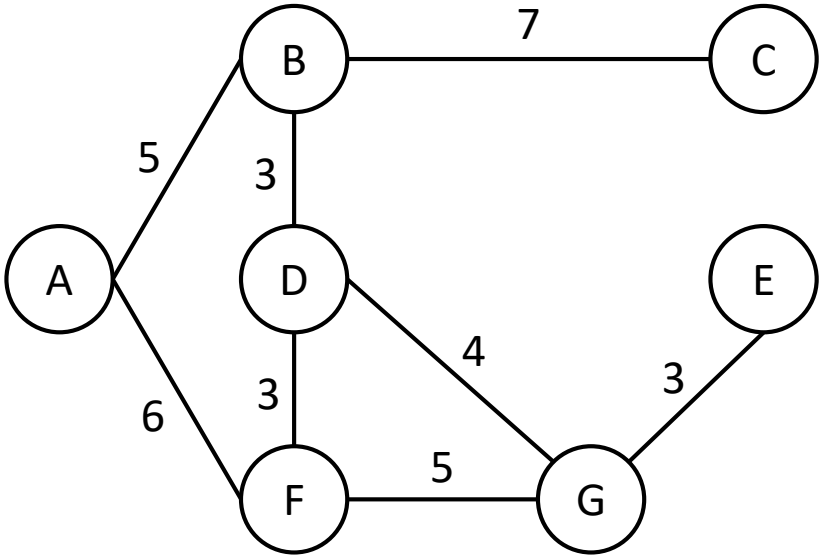


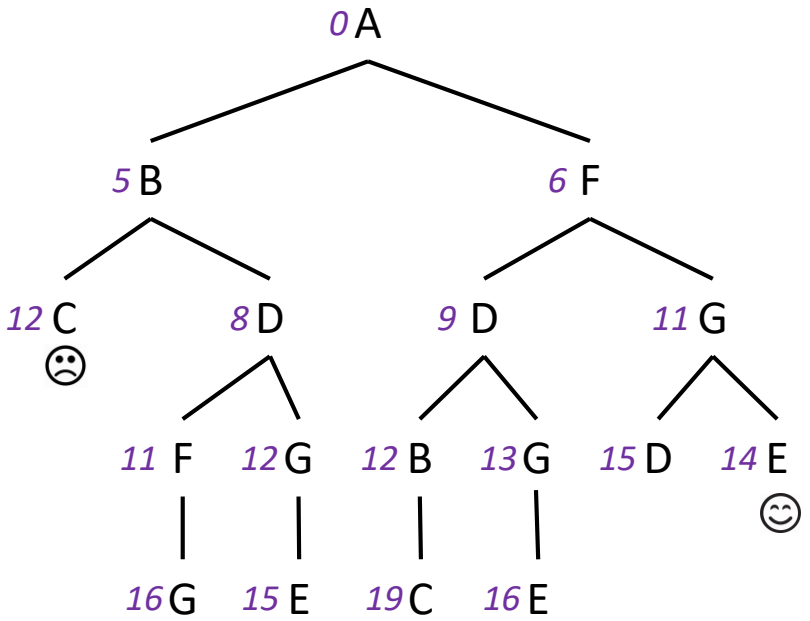
- Idea: discard a newly generated node if already present somewhere on the tree, we can do this with an **enqueued list**

Search for the optimal solution

- Now we assume to be interested in the solution with minimum cost (not just any path to the goal, but the cheapest possible)
- To devise an optimal search algorithm we take the moves from BFS. Why it seems reasonable to do that?
- We generalize the idea of BFS to that of Uniform Cost Search (UCS)
- BFS proceeds by *depth* levels, UCS does that by *cost* levels (as a consequence, if costs are all equal to some constant BFS and UCS coincide)
- Cost accumulated on a path from the start node to v: $g(v)$ (we should include a dependency on the path, but it will always be clear from the context)
- For now let's remove the enqueued list and the goal checking as we know it

Uniform Cost Search (UCS)





- Have we found the optimal path to the goal? In this problem instance, we can answer *yes* by inspecting the graph
- How about larger instances? Can we prove optimality?
- Actually, we can prove a stronger claim: every time UCS selects **for the first time** a node for expansion, the associated path leading to that node has minimum cost

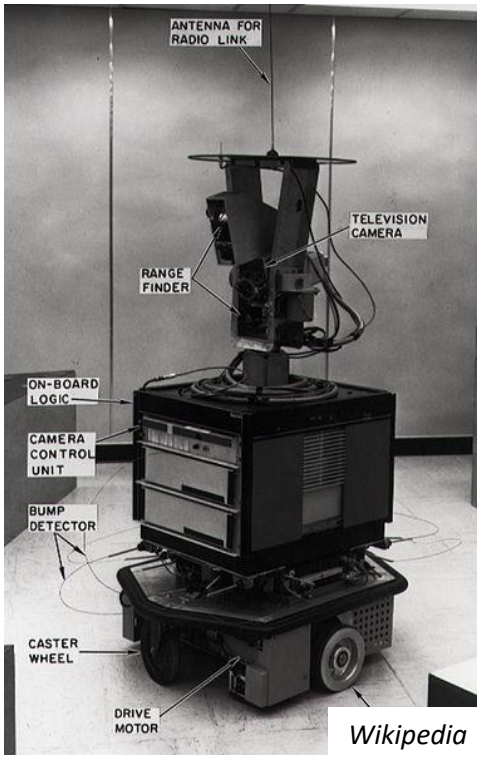
Informed vs non-informed search

- Besides its own rules, any search algorithm decides where to search next by leveraging some knowledge
- **Non-informed** search uses only knowledge specified at problem-definition time (e.g., goal and start nodes, edge costs), just like we saw in the previous examples
- An **informed** search might go beyond such knowledge
- Idea: using an estimate of how far a given node is from the goal
- Such an estimate is often called a **heuristic**

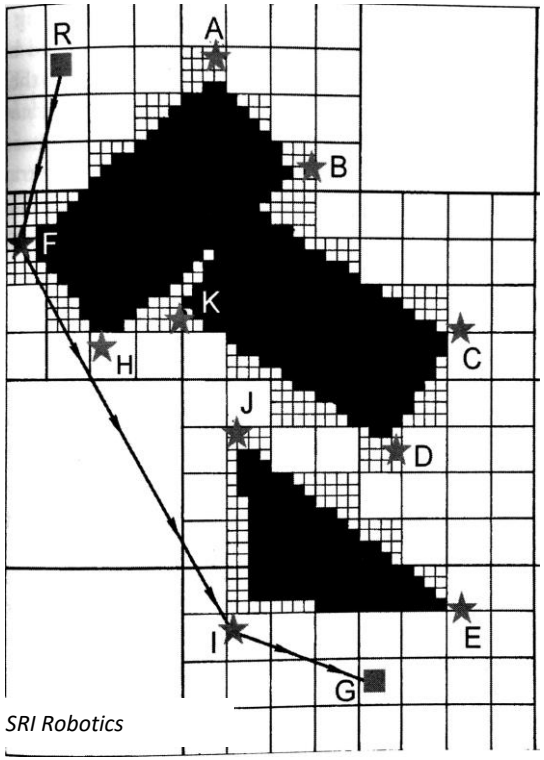
Estimate of the cost of the optimal path from node v to the goal: $h(v)$

A*

- The informed version of UCS is called A*
- Very popular search algorithm
- It was born in the early days of mobile robotics when, in 1968, Nilsson, Hart, and Raphael had to face a practical problem with Shakey (one of the ancestors of today's mobile robots)



Wikipedia



SRI Robotics

A*

- The idea behind A* is simple: perform a UCS, but instead of considering accumulated costs consider the following:

Heuristic
("cost-to-go")

$$f(n) = g(n) + h(n)$$

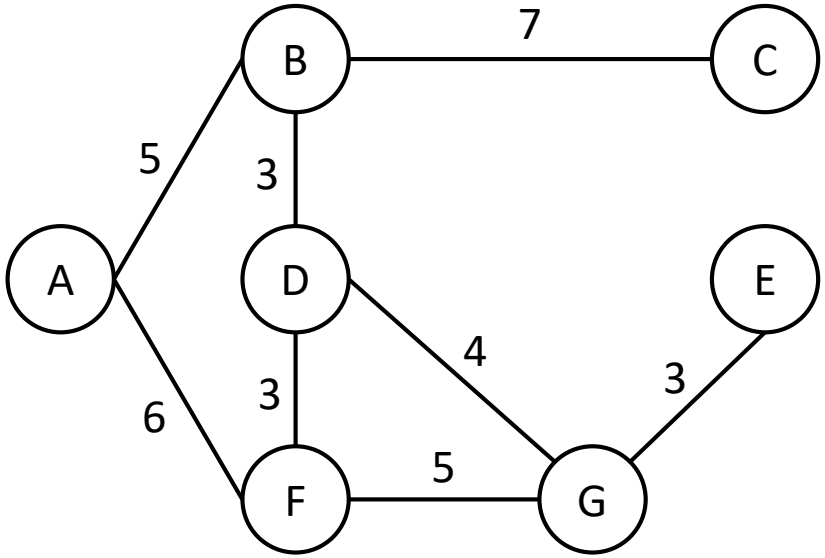
Cost accumulated
on the path to n
("cost-to-come")

- To guarantee that the search is sound and complete we need to require that the heuristic is **admissible**: it is an optimistic estimate or, more formally:

$$h(n) \leq \text{ Cost of the minimum path from n to the goal}$$

- If the heuristic is not admissible we might discard a path that could actually turn out to be better that the best candidate found so far

A*



node v	$h(v)$
A	10
B	7
C	1
D	3
E	0
F	7
G	2