Proof-search in Natural Deduction calculus for classical propositional logic

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Motivations

- The consensus is that natural deduction calculi are not suitable for proof-search because they lack the "deep symmetries" characterizing sequent calculi.
- Proof-search strategies to build natural deduction derivations are presented in:
  - W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). *Studia Logica*, 1998.
  - W. Sieg and S. Cittadini. Normal natural deduction proofs (in non-classical logics). *LNCS*, 2005.

  But these strategies are highly inefficient.
- It seems that that the only effective way to build derivations in natural deduction calculi consists in translating tableaux/sequent proofs.

Our contribution

We show that proof-search in natural deduction calculus for **Cl** (Propositional Classical Logic) can be efficiently performed.

In particular:

- we introduce **Ncr**, a variant of the usual natural deduction calculus for **Cl**
- we describe a proof-search procedure for **Ncr** not requiring backtracking nor loop-checking.
- Main related work:
  - W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). *Studia Logica*, 1998.
  - D.M. Gabbay and N. Olivetti. *Goal-Directed Proof Theory*. 2000. (in particular, the chapter devoted to goal-oriented proof-search for classical logic)

Natural Deduction calculus

The natural deduction calculus has been introduced to capture logical mathematical reasoning.

> *The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.*
>
> *I intended, first of all, to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).*
>
> *[Gentzen, "Investigations into logical deduction", 1934]*

Natural Deduction calculus

- Formulas A, B, ... of **Cl** are built starting from a set $\mathcal{V}$ of propositional variables:

$$\begin{array}{lcl} A, B & ::= & \bot \mid p \mid A \wedge B \mid A \vee B \mid A \rightarrow B \qquad p \in \mathcal{V} \\ \neg A & ::= & A \rightarrow \bot \end{array}$$

Natural Deduction calculus

- Formulas A, B, $\ldots$ of **Cl** are built starting from a set $\mathcal{V}$ of propositional variables:

$$
\begin{array}{lcl}
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\neg A & ::= & A \rightarrow \bot
\end{array}
$$

- For each logical connective it is defined an introduction rule (I-rule) and an elimination rule (E-rule)
  - I-rule
    How to introduce a compound formula.
    Infer a complex formula from already established components
  - E-rule
    How to de-construct information about a compound formula.
    Specify how components of assumed or established complex formulas can be used as arguments.

Localizing Hypothesis

A derivation $\mathcal{D}$ of B having open assumptions $A_1,\ldots,A_n$ is represented by a tree of the form

$$\begin{array}{c} A_1,\ldots,A_n \\ \mathcal{D} \\ B \end{array}$$

built according to the rules of the calculus.

In our presentation, it is more convenient to *localize hypothesis*

$$\begin{array}{c} \mathcal{D} \\ \Gamma \vdash B \end{array}$$

The context Γ contains the assumptions $A_1,\ldots,A_n$ on which B depends.

NK : Natural Deduction calculus for **CI** in sequent style

The calculus **NK**

$$\frac{}{A, \Gamma \vdash A}\ \mathrm{Id} \qquad \frac{\neg A, \Gamma \vdash \bot}{\Gamma \vdash A}\ \bot E_{\mathrm{C}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge I \qquad \frac{\Gamma \vdash A_0 \wedge A_1}{\Gamma \vdash A_k}\ \wedge E_k \quad k \in \{0, 1\}$$

$$\frac{\Gamma \vdash A_k}{\Gamma \vdash A_0 \vee A_1}\ \vee I_k \qquad \frac{\Gamma \vdash A \vee B \qquad A, \Gamma \vdash C \qquad B, \Gamma \vdash C}{\Gamma \vdash C}\ \vee E$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I \qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow E$$

The calculus **NK**

$$\frac{}{A, \Gamma \vdash A}\ \mathrm{Id} \qquad \frac{\neg A, \Gamma \vdash \bot}{\Gamma \vdash A}\ \bot E_{\mathrm{C}}$$

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$$\frac{\Gamma \vdash A_k}{\Gamma \vdash A_0 \vee A_1}\ \vee I_k \qquad \frac{\Gamma \vdash A \vee B \qquad A, \Gamma \vdash C \qquad B, \Gamma \vdash C}{\Gamma \vdash C}\ \vee E$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ \to I \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to E$$

Theorem (Completeness of NK)

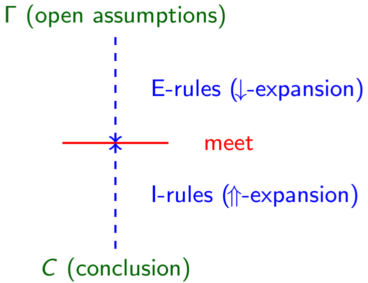
*$A \in$ **Cl** iff there exists a derivation of $\vdash A$ in* **NK**

A naïve proof-search strategy for **NK**

To perform proof-search, the basic idea is to *orient* rules application:

- Apply I-rules bottom-up ($\Uparrow$-expansion)
- Apply the E-rules top-down ($\downarrow$-expansion)

To get a derivation, $\Uparrow$-expansion and $\downarrow$-expansion must *meet* in the middle



A naïve proof-search strategy for **NK**

To formalize the strategy, we introduce the judgments:

- $\Gamma \vdash A\Uparrow$

  The sequent $\Gamma \vdash A$ is obtained by $\Uparrow$-expansion

- $\Gamma \vdash A\downarrow$

  The sequent $\Gamma \vdash A$ is obtained by $\downarrow$-expansion
  (A has been *extracted* from the assumptions Γ).

F. Pfenning. Automated theorem proving. Lecture notes, 2004.

W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). Studia Logica, 1998.

R. Dyckhoff and L. Pinto. Cut-elimination and a permutation-free sequent calculus for intuitionistic logic. Studia Logica, 1998.

NK + arrows $\downarrow, \Uparrow$ = **Nc**

Rules of **Nc**

- Rules for $\Uparrow$-expansion (to be applied bottom-up)

$$\frac{\Gamma \vdash A\Uparrow \qquad \Gamma \vdash B\Uparrow}{\Gamma \vdash A \wedge B\Uparrow} \wedge I \qquad \frac{A, \Gamma \vdash B\Uparrow}{\Gamma \vdash A \to B\Uparrow} \to I$$

$$\frac{\Gamma \vdash A\Uparrow}{\Gamma \vdash A \vee B\Uparrow} \vee I_0 \qquad \frac{\Gamma \vdash B\Uparrow}{\Gamma \vdash A \vee B\Uparrow} \vee I_1$$

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$$\frac{\Gamma \vdash A\Uparrow}{\Gamma \vdash A \vee B\Uparrow} \vee I_0 \qquad \frac{\Gamma \vdash B\Uparrow}{\Gamma \vdash A \vee B\Uparrow} \vee I_1$$

- Rules for $\downarrow$-expansion (to be applied top-down)

$$\frac{\Gamma \vdash A \wedge B\downarrow}{\Gamma \vdash A\downarrow} \wedge E_0 \qquad \frac{\Gamma \vdash A \wedge B\downarrow}{\Gamma \vdash B\downarrow} \wedge E_1$$

$$\frac{\Gamma \vdash A \to B\downarrow \qquad \Gamma \vdash A\Uparrow}{\Gamma \vdash B\downarrow} \to E \qquad \frac{}{A, \Gamma \vdash A\downarrow} \mathrm{Id}$$

Note that the right-most premise of $\to E$ is an $\Uparrow$-sequent

Rules of **Nc**

To match $\Uparrow$-expansion with $\downarrow$-expansion we need:

- Coercion

$$\frac{\Gamma \vdash A\downarrow}{\Gamma \vdash A\Uparrow} \downarrow\Uparrow$$

We can assume that A is prime (namely, $A \in \mathcal{V} \cup \{\bot\}$)

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- (Classical) $\bot$-elimination

$$\frac{\neg A, \Gamma \vdash \bot\downarrow}{\Gamma \vdash A\Uparrow} \bot E_{\mathrm{C}}$$

  We can assume $A \in \mathcal{V}$ or A is a disjunction

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- $\vee$-elimination

$$\frac{\Gamma \vdash A \vee B\downarrow \qquad A, \Gamma \vdash C\Uparrow \qquad B, \Gamma \vdash C\Uparrow}{\Gamma \vdash C\Uparrow} \vee E$$

We can assume C prime or C is a disjunction (namely, $C = C_0 \vee C_1$).

The calculus **Nc**

$$\frac{}{A, \Gamma \vdash A{\downarrow}}\ \mathrm{Id} \qquad \frac{\Gamma \vdash A{\downarrow}}{\Gamma \vdash A{\Uparrow}}\ {\downarrow\!\Uparrow} \quad (\dagger) \qquad \frac{\neg A, \Gamma \vdash \bot{\downarrow}}{\Gamma \vdash A{\Uparrow}}\ \bot E_{\mathrm{C}} \quad (\dagger)$$

$$\frac{\Gamma \vdash A{\Uparrow} \qquad \Gamma \vdash B{\Uparrow}}{\Gamma \vdash A \wedge B{\Uparrow}}\ \wedge I \qquad \frac{\Gamma \vdash A_0 \wedge A_1{\downarrow}}{\Gamma \vdash A_k{\downarrow}}\ \wedge E_k$$

$$\frac{\Gamma \vdash A_k{\Uparrow}}{\Gamma \vdash A_0 \vee A_1{\Uparrow}}\ \vee I_k \qquad \frac{\Gamma \vdash A \vee B{\downarrow} \quad A, \Gamma \vdash C{\Uparrow} \quad B, \Gamma \vdash C{\Uparrow}}{\Gamma \vdash C{\Uparrow}}\ \vee E \quad (\dagger)$$

$$\frac{A, \Gamma \vdash B{\Uparrow}}{\Gamma \vdash A \to B{\Uparrow}} \to I \qquad \frac{\Gamma \vdash A \to B{\downarrow} \qquad \Gamma \vdash A{\Uparrow}}{\Gamma \vdash B{\downarrow}} \to E$$

($\dagger$) Assumptions

$$\begin{array}{lcl} {\downarrow\!\Uparrow} & : & A \in \mathcal{V} \cup \{\bot\} \\ \bot E_{\mathrm{C}} & : & A \in \mathcal{V} \text{ or } A = A_0 \vee A_1 \\ \vee E & : & C \in \mathcal{V} \cup \{\bot\} \text{ or } C = C_0 \vee C_1 \end{array}$$

The calculus **Nc**

Derivations in **Nc** are *by definition* in normal form.
Actually, **Nc**-derivations correspond to **NK**-derivations in normal form.

For instance a detour of the kind

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A,\Gamma\vdash B\end{matrix}}{\Gamma\vdash A\to B}\to I \qquad \begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash B}\to E \quad \rightsquigarrow \quad \cfrac{\cfrac{\begin{matrix}[A],\,\Gamma\\ \vdots\\ B\end{matrix}}{A\to B}\to I \qquad \begin{matrix}\Gamma\\ \vdots\\ B\end{matrix}}{B}\to E$$

with a maximal formula $A\to B$ is not allowed in **Nc**.

$$\cfrac{\begin{matrix}\cancel{A,\Gamma\vdash B\downarrow}\\ \hdashline \Gamma\vdash A\to B\downarrow\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma\vdash A\Uparrow\end{matrix}}{\Gamma\vdash B\downarrow}\to E$$

$A\to B$ cannot be introduced in $\downarrow$-expansion!

The calculus **Nc**

Coercion rule

$$\frac{\Gamma \vdash A\downarrow}{\Gamma \vdash A\Uparrow}\downarrow\Uparrow$$

is crucial to "coerce" derivations in normal form.
To simulate **NK**-derivations in **Nc**, we need the converse of coercion

$$\frac{\Gamma \vdash A\Uparrow}{\Gamma \vdash A\downarrow}\Uparrow\downarrow$$

which allows one to build non-normal derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{A, \Gamma \vdash B\Uparrow}}{\Gamma \vdash A \to B\Uparrow}\to I}{\Gamma \vdash A \to B\downarrow}\Uparrow\downarrow \qquad \dfrac{\cdots}{\Gamma \vdash A\Uparrow}}{\Gamma \vdash B\downarrow}\to E$$

Theorem (Normalization of NK)

$\Gamma \vdash A$ *is provable in* **NK** *iff* $\Gamma \vdash A\Uparrow$ *is provable in* **Nc**.

Proof-search strategy for **Nc**

We alternate $\Uparrow$-expansion and $\downarrow$-expansion phases.

(1) $\Uparrow$-expansion

To prove $\Gamma \vdash A \Uparrow$, backward apply introduction rules.

We stop whenever we get a leaf sequent $\Gamma' \vdash K \Uparrow$ such that K is prime or a disjunction

$$\begin{array}{ccc} \Gamma'_1 \vdash K_1 \Uparrow & \dots & \Gamma'_n \vdash K_n \Uparrow \\ & \vdots & \\ & \Gamma \vdash A \Uparrow & \end{array}$$

Now, we have to expand the leaves

Proof-search strategy for **Nc**

We have three possible ways to expand a leaf $\Gamma' \vdash K \Uparrow$.

Proof-search strategy for **Nc**

We have three possible ways to expand a leaf $\Gamma' \vdash K \Uparrow$.

(1.1) If K is prime, we can apply coercion.

$$\cfrac{\Gamma' \vdash K \downarrow}{\Gamma' \vdash K \Uparrow} \downarrow\Uparrow$$

$$\vdots$$

$$\Gamma \vdash A \Uparrow$$

Proof-search strategy for **Nc**

We have three possible ways to expand a leaf $\Gamma' \vdash K{\Uparrow}$.

(1.1) If K is prime, we can apply coercion.

$$\dfrac{\Gamma' \vdash K{\downarrow}}{\Gamma' \vdash K{\Uparrow}} \Downarrow\!\Uparrow$$

$$\vdots$$

$$\Gamma \vdash A{\Uparrow}$$

(1.2) If K is a propositional variable or a disjunction, we can apply classical $\bot$-elimination.

$$\dfrac{\neg K, \Gamma' \vdash \bot{\downarrow}}{\Gamma' \vdash K{\Uparrow}} \bot E_{\mathrm{C}}$$

$$\vdots$$

$$\Gamma \vdash A{\Uparrow}$$

Proof-search strategy for **Nc**

(1.3) If K is prime or a disjunction, we can apply $\vee$-elimination.

$$\dfrac{\Gamma' \vdash D_0 \vee D_2 \downarrow \qquad D_0, \Gamma' \vdash K \Uparrow \qquad D_1, \Gamma' \vdash K \Uparrow}{\Gamma' \vdash K \Uparrow} \vee E$$

$$\vdots$$

$$\Gamma \vdash A \Uparrow$$

Proof-search strategy for **Nc**

(1.3) If K is prime or a disjunction, we can apply $\vee$-elimination.

$$\dfrac{\Gamma' \vdash D_0 \vee D_2 \downarrow \qquad D_0, \Gamma' \vdash K \Uparrow \qquad D_1, \Gamma' \vdash K \Uparrow}{\Gamma' \vdash K \Uparrow} \vee E$$

$$\vdots$$

$$\Gamma \vdash A \Uparrow$$

In all cases, we generate new leaves that must be proved.

- To prove an $\Uparrow$-sequent, we continue the current $\Uparrow$-expansion phase.
- To prove a $\downarrow$-sequent, we start a new $\downarrow$-expansion phase.

Proof-search strategy for **Nc**

(2) $\downarrow$-expansion

To prove $\Gamma \vdash K\downarrow$:

- Select $H \in \Gamma$ (head formula)
- Starting from the axiom sequent

$$\Gamma \vdash H\downarrow$$

apply $\wedge, \rightarrow$-elimination rules with the goal to extract K from H.

$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash H\downarrow}\,\mathrm{Id}}{\Gamma \vdash H_1\downarrow}\,\mathcal{R}_1}{\Gamma \vdash H_2\downarrow}\,\mathcal{R}_2 \qquad \begin{array}{l} H \in \Gamma \\ \mathcal{R}_1, \mathcal{R}_2 \cdots \in \{\wedge E_k, \rightarrow E\} \end{array}$$

$$\cdots$$

$$\Gamma \vdash K\downarrow$$

Proof-search strategy for **Nc**

$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash H\downarrow}}{\Gamma \vdash H_1\downarrow}}{\Gamma \vdash H_2\downarrow} \quad H \in \Gamma$$

$$\cdots$$

$$\Gamma \vdash K\downarrow$$

The formulas $H_1, H_2, \ldots, K$ obtained in the right are subformulas of H of a special form, we call them strictly positive subformula of H.

Formally:

- $\mathrm{Sf}^+(H)$: the set of strictly positive subformula of H.
- $Q \in \mathrm{Sf}^+(H)$ iff:

$$Q \quad ::= \quad H \mid Q' \wedge A \mid A \wedge Q' \mid A \rightarrow Q'$$

where $Q' \in \mathrm{Sf}^+(H)$ and A is any formula.

Proof-search strategy for **Nc**

To narrow the search space, we refine $\downarrow$-expansion:

(2') $\downarrow$-expansion

To prove $\Gamma \vdash K\downarrow$:

- Select $H \in \Gamma$ (head formula) *such that* $K \in \mathrm{Sf}^{+}(H)$.
- Starting from the axiom sequent $\Gamma \vdash H\downarrow$ apply $\wedge, \rightarrow$-elimination rules with the goal to extract K from H.

$$\dfrac{}{\Gamma \vdash H\downarrow}\,\mathrm{Id} \qquad\qquad H \in \Gamma \textit{ such that } K \in \mathrm{Sf}^{+}(H)$$
$$\ldots$$
$$\Gamma \vdash K\downarrow$$

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- Starting from the axiom sequent $\Gamma \vdash H\downarrow$ apply $\wedge, \to$-elimination rules with the goal to extract K from H.

$$\dfrac{}{\Gamma \vdash H\downarrow}\ \mathrm{Id} \qquad H \in \Gamma \textit{ such that } K \in \mathrm{Sf}^+(H)$$
$$\cdots$$
$$\Gamma \vdash K\downarrow$$

Note that $\to E$ generates a new $\Uparrow$-sequent, which must be $\Uparrow$-expanded.

$$\dfrac{\dfrac{}{\Gamma \vdash H\downarrow}\ \mathrm{Id} \quad \cdots \quad \Gamma \vdash A \to B\downarrow \qquad \Gamma \vdash A\Uparrow}{\Gamma \vdash B\downarrow}\ \to E$$
$$\cdots$$
$$\Gamma \vdash K\downarrow$$

A proof-search example

Let us search for a **Nc**-derivation of $p \vee \neg p$

We start an $\Uparrow$-expansion phase from the sequent:

$$\vdash p \vee \neg p \Uparrow$$

A proof-search example

Let us search for a **Nc**-derivation of $p \vee \neg p$

We start an $\Uparrow$-expansion phase from the sequent:

$$\vdash p \vee \neg p \Uparrow$$

We have now three choices:

$$\frac{\vdash p \Uparrow}{\vdash p \vee \neg p \Uparrow} \vee I_0 \qquad \frac{\vdash \neg p \Uparrow}{\vdash p \vee \neg p \Uparrow} \vee I_1 \qquad \frac{\neg(p \vee \neg p) \vdash \bot \Downarrow}{\vdash p \vee \neg p \Uparrow} \bot E_{\mathrm{C}}$$

(1) Apply $\vee I_0$ (2) Apply $\vee I_1$ (3) Apply $\bot E_{\mathrm{C}}$

A proof-search example

(1) Let us apply $\vee I_0$. We have two choices:

$$\dfrac{\dfrac{\vdash p\downarrow}{\vdash p\Uparrow}\,\Downarrow\!\Uparrow}{\vdash p \vee \neg p\Uparrow}\,\vee I_0 \qquad\qquad \dfrac{\dfrac{\neg p \vdash \bot\downarrow}{\vdash p\Uparrow}\,\bot E_{\mathrm{C}}}{\vdash p \vee \neg p\Uparrow}\,\vee I_0$$

(1.1) Apply $\Downarrow\!\Uparrow$
Fail

(1.2) Apply $\bot E_{\mathrm{C}}$

A proof-search example

(1.2)

$$\cfrac{\cfrac{\neg p \vdash \bot\downarrow}{\vdash p\Uparrow}\ \bot E_{\mathrm{C}}}{\vdash p \vee \neg p\Uparrow}\ \vee I_0$$

We can extract $\bot$ from $\neg p$ starting a $\downarrow$-phase from the axiom sequent

$$\neg p \vdash \neg p\downarrow$$

and applying $\to E$. We get

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\neg p \vdash \neg p\downarrow}\ \mathrm{Id} \qquad \neg p \vdash p\Uparrow}{\neg p \vdash \bot\downarrow}\ \to E}{\vdash p\Uparrow}\ \bot E_{\mathrm{C}}}{\vdash p \vee \neg p\Uparrow}\ \vee I_0$$

A proof-search example

We can $\Uparrow$-expand the leaf $\neg p \vdash p\Uparrow$ in two ways:

$$\dfrac{\neg p \vdash p\Downarrow}{\neg p \vdash p\Uparrow}\Downarrow\!\Uparrow \qquad\qquad \dfrac{\neg p \vdash \bot\Downarrow}{\neg p \vdash p\Uparrow}\bot E_{\mathrm{C}}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\vdash p \vee \neg p\Uparrow \qquad\qquad \vdash p \vee \neg p\Uparrow$$

(1.2.1) Apply $\Downarrow\!\Uparrow$
Fail

(1.2.2) Apply $\bot E_{\mathrm{C}}$
Loop

$$\dfrac{\dfrac{\dfrac{}{\neg p \vdash \neg p\Downarrow}\mathrm{Id} \qquad \neg p \vdash p\Uparrow}{\neg p \vdash \bot\Downarrow}\to E}{\neg p \vdash p\Uparrow}\bot E_{\mathrm{C}}$$

$$\vdots$$

$$\vdash p \vee \neg p\Uparrow$$

A proof-search example

We have to backtrack and try (2), namely $\vee I_1$
After some expansion step, we get

$$\dfrac{\dfrac{p \vdash \bot \Uparrow}{\vdash \neg p \Uparrow} \rightarrow I}{\vdash p \vee \neg p \Uparrow} \vee I_1$$

and proof-search fails.
We need to backtrack once again.

A proof-search example

It remains to try (3), namely $\bot E_{\mathrm{C}}$.
We get:

$$\dfrac{\dfrac{}{\neg(p \vee \neg p) \vdash \neg(p \vee \neg p){\downarrow}}\,\mathrm{Id} \qquad \neg(p \vee \neg p) \vdash p \vee \neg p{\Uparrow}}{\dfrac{\neg(p \vee \neg p) \vdash \bot{\downarrow}}{\vdash p \vee \neg p{\Uparrow}}\,\bot E_{\mathrm{C}}}\,{\to} E$$

A proof-search example

Now, we have three possible choices

$$\frac{\neg(p \vee \neg p) \vdash p \Uparrow}{\neg(p \vee \neg p) \vdash p \vee \neg p \Uparrow} \vee I_0$$

$$\vdots$$

$$\vdash p \vee \neg p \Uparrow$$

(3.1) Apply $\vee I_0$

$$\frac{\neg(p \vee \neg p) \vdash \neg p \Uparrow}{\neg(p \vee \neg p) \vdash p \vee \neg p \Uparrow} \vee I_1$$

$$\vdots$$

$$\vdash p \vee \neg p \Uparrow$$

(3.2) Apply $\vee I_1$

$$\frac{\neg(p \vee \neg p) \vdash \bot \Downarrow}{\neg(p \vee \neg p) \vdash p \vee \neg p \Uparrow} \bot E_{\mathrm{C}}$$

$$\vdots$$

$$\vdash p \vee \neg p \Uparrow$$

(3.3) Apply $\bot E_{\mathrm{C}}$

A proof-search example

All the three ways lead to a successful derivation, possibly with some redundancies.

The most concise derivation corresponds to choice (3.2)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p,\neg(p\vee\neg p)\vdash\neg(p\vee\neg p)\downarrow}\,\mathrm{Id} \qquad \dfrac{\dfrac{\dfrac{}{p,\neg(p\vee\neg p)\vdash p\downarrow}\,\mathrm{Id}}{p,\neg(p\vee\neg p)\vdash p\Uparrow}\,\downarrow\Uparrow}{p,\neg(p\vee\neg p)\vdash p\vee\neg p\Uparrow}\,\vee I_0}{p,\neg(p\vee\neg p)\vdash\bot\Uparrow}}{\neg(p\vee\neg p)\vdash\neg p\Uparrow}\,\to I}{\neg(p\vee\neg p)\vdash p\vee\neg p\Uparrow}\,\vee I_1$$

$$\vdots$$

$$\vdash p\vee\neg p\Uparrow$$

This corresponds to the standard derivation in normal form of $p \vee \neg p$.

A proof-search example

Compare with the sequent derivation of the same formula $p \vee \neg p$:

$$\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,\text{Ax}}{\Rightarrow p, \neg p}\,R\rightarrow}{\Rightarrow p \vee \neg p}\,R\vee$$

- very compact and plain
- no choice points
- no backtracking
- sequents are decreasing hence branches have finite length (termination)

Proof-search strategy for **Nc**

This naïve strategy suffers from the huge search space:

- Contexts never decrease, hence an assumption might be used more and more times
- too many backtrack points
- some mechanism is needed to guarantee termination (e.g., loop-checking).

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This is in disagreement with the proof-search strategies based on standard sequent/tableaux calculi for **CI**, where:

- a formula occurrence can be used at most once along a branch
- no backtracking is needed
- termination is guaranteed by the fact that at each step at least a formula is decomposed.

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- a formula occurrence can be used at most once along a branch
- no backtracking is needed
- termination is guaranteed by the fact that at each step at least a formula is decomposed.

Can we recover these nice properties
in natural deduction proof-search?

On assumptions control

- An application of $\bot E_{\mathrm{C}}$ transfers the current right-formula A on the left, negating it:

$$\frac{\neg A, \Gamma \vdash \bot\downarrow}{\Gamma \vdash A\Uparrow}\ \bot E_{\mathrm{C}}$$

Note that this breaks the strict subformula property
From now on, the assumption $\neg A$ cannot be thrown down.

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  Note that this breaks the strict subformula property
  From now on, the assumption $\neg A$ cannot be thrown down.

- Using assumption $\neg A$ and $\bot E_{\mathrm{C}}$, we can regain A on the right:

$$\cfrac{\cfrac{\cfrac{}{\neg A, \neg B, \Gamma_1 \vdash \neg A\downarrow}\ \mathrm{Id} \qquad \neg A, \neg B, \Gamma_1 \vdash A\Uparrow}{\neg A, \neg B, \Gamma_1 \vdash \bot\downarrow}\ \to E}{\neg A, \Gamma_1 \vdash B\Uparrow}\ \bot E_{\mathrm{C}}$$

$$\vdots$$

$$\frac{\neg A, \Gamma \vdash \bot\downarrow}{\Gamma \vdash A\Uparrow}\ \bot E_{\mathrm{C}}$$

On assumptions control

- By repeatedly applying this pattern, we get an infinite branch where the right formula A can be used as many times we want:

$$\dfrac{\dfrac{\dfrac{\vdots}{\neg A, \Gamma_2 \vdash A\Uparrow}}{\vdots \atop \neg A, \Gamma_1 \vdash A\Uparrow} \atop \vdots \atop \neg A, \Gamma \vdash \bot\Downarrow}{\Gamma \vdash A\Uparrow}\ \bot E_{\mathrm{C}} \qquad \Gamma \subseteq \Gamma_1 \subseteq \Gamma_2$$

To get the same effect but in a more controlled way:

replace $\bot E_{\mathrm{C}}$ with restart rule [Gabbay&Olivetti,2000]

Restart

Restart rule allows one to restart from a previous right-formula:

$$\begin{array}{c} \dots \\ \dfrac{\Gamma_1 \vdash A \Uparrow}{\Gamma' \vdash B \Uparrow} \text{ Restart from } A \\ \dots \\ \Gamma \vdash A \Uparrow \\ \dots \end{array}$$

- We apply Restart is in $\Uparrow$-expansion, if the current right formula is prime
- Formulas usable for restart are stored in a restart set Δ

$$\dfrac{\Gamma \vdash A \Uparrow \qquad \Delta ::= F, \Delta'}{\Gamma \vdash F \Uparrow \qquad \Delta ::= A, \Delta'} \text{ Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

restart from A and store F in Δ

Restart

Restart rule allows one to restart from a previous right-formula:

$$\dfrac{\begin{array}{c}\dots\\ \Gamma_1 \vdash A \Uparrow\end{array}}{\begin{array}{c}\Gamma' \vdash B \Uparrow\\ \dots\\ \Gamma \vdash A \Uparrow\\ \dots\end{array}} \text{ Restart from } A$$

- We apply Restart is in $\Uparrow$-expansion, if the current right formula is prime
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$$\frac{\Gamma \vdash A \Uparrow \qquad \Delta \ ::= \ F, \Delta'}{\Gamma \vdash F \Uparrow \qquad \Delta \ ::= \ A, \Delta'} \text{ Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

restart from A and store F in Δ

This leads to the natural deduction calculus **Ncr** (**Nc** with restart)

The calculus **Ncr**

$$\textbf{Ncr} = \textbf{Nc} \begin{array}{ll} - & \text{classical } \bot\text{-elimination } \bot E_{\mathrm{C}} \\ + & \text{restart} \\ + & \text{intuitionistic } \bot\text{-elimination } \bot E_{\mathrm{I}} \end{array}$$

The calculus **Ncr**

$$\mathbf{Ncr} \;=\; \mathbf{Nc} \quad \begin{array}{ll} - & \text{classical } \bot\text{-elimination } \bot E_{\mathrm{C}} \\ + & \text{restart} \\ + & \text{intuitionistic } \bot\text{-elimination } \bot E_{\mathrm{I}} \end{array}$$

Sequents need more structure:

$\Uparrow$-sequent: $\Gamma \vdash A \Uparrow ; \Delta$

logical meaning: $\bigwedge \Gamma \rightarrow (A \vee \bigvee \Delta)$

- Γ: set of assumptions
- A: right-formula (the formula to be proved)
- Δ: restart set (formulas available for restart)

Proof-search starts with an empty restart set.

The calculus **Ncr**

- Restart (to be improved to avoid loops)

$$\frac{\Gamma \vdash A\Uparrow;\ F, \Delta}{\Gamma \vdash F\Uparrow;\ A, \Delta}\ \text{Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

The calculus **Ncr**

- Restart (to be improved to avoid loops)

$$\frac{\Gamma \vdash A\Uparrow;\ F, \Delta}{\Gamma \vdash F\Uparrow;\ A, \Delta}\ \text{Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

- $\wedge$-introduction

$$\frac{\Gamma \vdash A\Uparrow;\ \Delta \qquad \Gamma \vdash B\Uparrow;\ \Delta}{\Gamma \vdash A \wedge B\Uparrow;\ \Delta}\ \wedge I$$

The calculus **Ncr**

- Restart (to be improved to avoid loops)

$$\frac{\Gamma \vdash A\Uparrow;\, F, \Delta}{\Gamma \vdash F\Uparrow;\, A, \Delta}\ \text{Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

- $\wedge$-introduction

$$\frac{\Gamma \vdash A\Uparrow;\, \Delta \qquad \Gamma \vdash B\Uparrow;\, \Delta}{\Gamma \vdash A \wedge B\Uparrow;\, \Delta}\ \wedge I$$

- $\vee$-introduction

$$\frac{\Gamma \vdash A\Uparrow;\, B, \Delta}{\Gamma \vdash A \vee B\Uparrow;\, \Delta}\ \vee I$$

Note that we need only one rule, which retains the first disjunct. The second one can be recovered by restart.

The calculus **Ncr**

- Restart (to be improved to avoid loops)

$$\frac{\Gamma \vdash A\Uparrow;\, F, \Delta}{\Gamma \vdash F\Uparrow;\, A, \Delta}\ \text{Restart} \qquad F \in \mathcal{V} \cup \{\bot\}$$

- $\wedge$-introduction

$$\frac{\Gamma \vdash A\Uparrow;\, \Delta \qquad \Gamma \vdash B\Uparrow;\, \Delta}{\Gamma \vdash A \wedge B\Uparrow;\, \Delta}\ \wedge I$$

- $\vee$-introduction

$$\frac{\Gamma \vdash A\Uparrow;\, B, \Delta}{\Gamma \vdash A \vee B\Uparrow;\, \Delta}\ \vee I$$

  Note that we need only one rule, which retains the first disjunct. The second one can be recovered by restart.

- $\rightarrow$-introduction

$$\frac{A, \Gamma \vdash B\Uparrow;\, \Delta}{\Gamma \vdash A \rightarrow B\Uparrow;\, \Delta}\ \rightarrow I$$

On resource consumption

Before continuing the presentation of **Ncr**, we point out another issue on proof-search in **Nc** (not solved by restart).

Let us consider a derivation of

$$p \wedge (p \rightarrow q) \ \rightarrow \ q$$

in the classical sequent calculus (**G3**-style):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, p \rightarrow q \Rightarrow p}\,\text{Ax} \qquad \dfrac{}{p, q \Rightarrow q}\,\text{Ax}}{p, p \rightarrow q \Rightarrow q}\,L\rightarrow}{p \wedge (p \rightarrow q) \Rightarrow q}\,L\wedge}{\Rightarrow p \wedge (p \rightarrow q) \ \rightarrow \ q}\,R\rightarrow$$

The assumption $p \wedge (p \rightarrow q)$ is used once

On resource consumption

In contrast, to prove the same formula in **Nc**, we have to use the assumption $p \wedge (p \to q)$ twice.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \wedge (p \to q) \vdash p \wedge (p \to q){\downarrow}}\,\mathrm{Id}}{p \wedge (p \to q) \vdash p \to q{\downarrow}}\,\wedge E_1 \qquad \dfrac{\dfrac{\dfrac{}{p \wedge (p \to q) \vdash p \wedge (p \to q){\downarrow}}\,\mathrm{Id}}{p \wedge (p \to q) \vdash p{\downarrow}}\,\wedge E_0}{p \wedge (p \to q) \vdash p{\Uparrow}}\,{\downarrow\Uparrow}}{p \wedge (p \to q) \vdash q{\downarrow}}\,\to E}{p \wedge (p \to q) \vdash q{\Uparrow}}\,{\downarrow\Uparrow}}{\vdash (p \wedge (p \to q)) \;\to\; q{\Uparrow}}\,\to I
$$

Compare with the sequent derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{p, p \to q \Rightarrow p}\,\mathrm{Ax} \qquad \dfrac{}{p, q \Rightarrow q}\,\mathrm{Ax}}{p, p \to q \Rightarrow q}\,L\to}{p \wedge (p \to q) \Rightarrow q}\,L\wedge}{\Rightarrow p \wedge (p \to q) \;\to\; q}\,R\to
$$

On resource consumption

This is due to the different behaviour in managing an assumption $A \wedge B$:

- Sequent calculus

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \; L\wedge$$

Both the conjuncts A and B are retained on the left and are available as assumptions.

- **Nc**

$$\frac{\Gamma \vdash A \wedge B{\downarrow}}{\Gamma \vdash A{\downarrow}} \; \wedge E_0 \qquad \frac{\Gamma \vdash A \wedge B{\downarrow}}{\Gamma \vdash B{\downarrow}} \; \wedge E_1$$

In both cases, one between the conjuncts A and B is lost.

To regain it, we need to re-prove $\Gamma \vdash A \wedge B{\downarrow}$, and this introduces some overhead in proof-search.

On resource consumption

To overcome the problem, in $\downarrow$-expansion we do not throw down the unused conjuncts, but we preserve them exploiting a resource set Θ.

On resource consumption

To overcome the problem, in $\downarrow$-expansion we do not throw down the unused conjuncts, but we preserve them exploiting a resource set Θ.

- At the beginning of a $\downarrow$-expansion phase, Θ is empty

$$\frac{}{\Gamma \vdash H{\downarrow} \qquad \Theta \ ::= \ \emptyset}\,\mathrm{Id} \qquad H \in \Gamma$$

On resource consumption

To overcome the problem, in $\downarrow$-expansion we do not throw down the unused conjuncts, but we preserve them exploiting a resource set Θ.

- At the beginning of a $\downarrow$-expansion phase, Θ is empty

$$\frac{}{\Gamma \vdash H\downarrow \qquad \Theta ::= \emptyset}\,\mathrm{Id} \qquad H \in \Gamma$$

- Whenever an $\wedge$-elimination rule is applied, Θ is updated by adding the unused conjunct

$$\frac{\Gamma \vdash A \wedge B\downarrow \qquad \Theta}{\Gamma \vdash A\downarrow \qquad \Theta \cup \{B\}}\,\wedge E_0 \qquad\qquad \frac{\Gamma \vdash A \wedge B\downarrow \qquad \Theta}{\Gamma \vdash B\downarrow \qquad \Theta \cup \{A\}}\,\wedge E_1$$

This is similar to the *LL(Local Linear)-computation* paradigm of [Gabbay&Olivetti,2000]

On resource consumption

To combine restart with LL-computation, $\downarrow$-sequents must be refined:

$\downarrow$-sequent: $\Gamma\,;\,H \vdash A\!\downarrow\,;\,\Delta\,;\,\Theta$

logical meaning: $(\bigwedge \Gamma \wedge H) \rightarrow ((A \wedge \bigwedge \Theta) \vee \bigvee \Delta)$

- $\Gamma \cup \{H\}$: available assumptions
- H (head formula): assumption selected at the beginning of $\downarrow$-expansion, to settle the initial axiom sequent
- A: right-formula (the formula to be proved)
- Δ: restart set (not used in $\downarrow$-expansion)
- Θ: the resource set (updated by $\wedge$-elimination applications)

On resource consumption

$\downarrow$-expansion starts from an axiom sequent with empty resource set

$$\Gamma\,;\,H \vdash H{\downarrow}\,;\,\Delta\,;\,\emptyset$$

By applying $\wedge, \rightarrow$-elimination rules, we get a branch of the form

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,H \vdash H{\downarrow}\,;\,\Delta\,;\,\emptyset}\,\mathrm{Id}}{\Gamma\,;\,H \vdash H_1{\downarrow}\,;\,\Delta\,;\,\Theta_1}}{\Gamma\,;\,H \vdash H_2{\downarrow}\,;\,\Delta\,;\,\Theta_2}$$

$$\vdots$$

We remark that:

- $H_1,\ H_2,\ \ldots\ \in \mathrm{Sf}^+(H)$ (the set of strictly pos. subformulas of H)
- $\Theta_1,\ \Theta_2,\ \ldots\ \subseteq \mathrm{Sf}^+(H)$

The calculus **Ncr**

- ∧-elimination

$$\frac{\Gamma\,;\,H \vdash A \wedge B{\downarrow}\,;\,\Delta\,;\,\Theta}{\Gamma\,;\,H \vdash A{\downarrow}\,;\,\Delta\,;\,B,\Theta}\ \wedge E_0 \qquad \frac{\Gamma\,;\,H \vdash A \wedge B{\downarrow}\,;\,\Delta\,;\,\Theta}{\Gamma\,;\,H \vdash B{\downarrow}\,;\,\Delta\,;\,A,\Theta}\ \wedge E_1$$

The calculus **Ncr**

- $\wedge$-elimination

$$\frac{\Gamma\,;\,H \vdash A \wedge B{\downarrow}\,;\,\Delta\,;\,\Theta}{\Gamma\,;\,H \vdash A{\downarrow}\,;\,\Delta\,;\,B, \Theta}\,\wedge E_0 \qquad \frac{\Gamma\,;\,H \vdash A \wedge B{\downarrow}\,;\,\Delta\,;\,\Theta}{\Gamma\,;\,H \vdash B{\downarrow}\,;\,\Delta\,;\,A, \Theta}\,\wedge E_1$$

- $\rightarrow$-elimination

$$\frac{\Gamma\,;\,H \vdash A \rightarrow B{\downarrow}\,;\,\Delta\,;\,\Theta \qquad \Gamma, \Theta \vdash A{\Uparrow}\,;\,\Delta}{\Gamma\,;\,H \vdash B{\downarrow}\,;\,\Delta\,;\,\Theta}\rightarrow E$$

The right-most premise starts a new $\Uparrow$-expansion phase where:

- the available assumptions are $\Gamma \cup \Theta$
- the assumption H is not usable any more, but it has been replaced by the formulas in Θ (which are strictly positive subformulas of H).

The calculus **Ncr**

- Coercion

  To prove $\Gamma \vdash p{\Uparrow} ; \Delta$ using coercion:

  - Non-deterministically select $H \in \Gamma$ such that $p \in \mathrm{Sf}^+(H)$
    [Non-deterministically = No backtracking !]
  - Start a $\Downarrow$-expansion phase from the axiom sequent

    $$\Gamma_H ; H \vdash H{\downarrow} ; p, \Delta ; \emptyset \qquad \Gamma_H = \Gamma \setminus \{H\}$$

    with the goal to extract p from H.
    Note that p has been added to the restart set.

$$\dfrac{}{\Gamma_H ; H \vdash H{\downarrow} ; p, \Delta ; \emptyset}\ \mathrm{Id}$$

$$\vdots$$

$$\Gamma_H ; H \vdash p{\downarrow} ; p, \Delta ; \Theta$$

To close the gap, coercion rule must have the form:

$$\dfrac{\Gamma_H ; H \vdash p{\downarrow} ; p, \Delta ; \Theta}{H, \Gamma \vdash p{\Uparrow} ; \Delta}\ {\downarrow\Uparrow}$$

The calculus **Ncr**

- Restart

  We split restart into two rules.

The calculus **Ncr**

- Restart

  We split restart into two rules.

  - $\mathrm{R_c}$
    Restart from a compound formula D, namely $D \notin \mathcal{V}$ and $D \neq \bot$.

$$\frac{\Gamma \vdash D\Uparrow;\ F, \Delta}{\Gamma \vdash F\Uparrow;\ D, \Delta}\ \mathrm{R_c} \qquad F \in \mathcal{V} \cup \{\bot\}$$

The calculus **Ncr**

- Restart

  We split restart into two rules.

  - $\mathrm{R_c}$

    Restart from a compound formula D, namely $D \notin \mathcal{V}$ and $D \neq \bot$.

$$\frac{\Gamma \vdash D\Uparrow;\ F, \Delta}{\Gamma \vdash F\Uparrow;\ D, \Delta}\ \mathrm{R_c} \qquad F \in \mathcal{V} \cup \{\bot\}$$

  - $\mathrm{R_p}$

    Restart from a propositional variable p and, to avoid infinite loops, immediately apply coercion:

$$\frac{\dfrac{\Gamma_H\,;\, H \vdash\ p\downarrow;\ F, p, \Delta\,;\,\Theta}{H, \Gamma \vdash\ p\Uparrow;\ F, \Delta}\ \downarrow\Uparrow}{H, \Gamma \vdash\ F\Uparrow;\ p, \Delta}\ \text{Restart} \qquad \begin{array}{l} \Gamma_H = \Gamma \setminus \{H\} \\ F \in \mathcal{V} \cup \{\bot\} \end{array}$$

More succinctly:

$$\frac{\Gamma_H\,;\, H \vdash\ p\downarrow;\ F, p, \Delta\,;\,\Theta}{H, \Gamma \vdash\ F\Uparrow;\ p, \Delta}\ \mathrm{R_p}$$

The calculus **Ncr**

- intuitionistic $\bot$-elimination

  To prove $\Gamma \vdash F\Uparrow ; \Delta$, with F prime, using $\bot$-elimination:

  - Non-deterministically select $H \in \Gamma$ such that $\bot \in \mathrm{Sf}^+(H)$
  - Start a $\downarrow$-expansion phase from the axiom sequent

  $$\Gamma_H ; H \vdash H\downarrow ; F, \Delta ; \emptyset \qquad \Gamma_H = \Gamma \setminus \{H\}$$

  with the goal to extract $\bot$ from H
  Note that F has been added to the restart set.

$$\dfrac{}{\Gamma_H ; H \vdash H\downarrow ; F, \Delta ; \emptyset}\ \mathrm{Id}$$

$$\vdots$$

$$\Gamma_H ; H \vdash \bot\downarrow ; F, \Delta ; \Theta$$

To close the gap, $\bot E_{\mathrm{I}}$ must have the form:

$$\dfrac{\Gamma_H ; H \vdash \bot\downarrow ; F, \Delta ; \Theta}{H, \Gamma \vdash F\Uparrow ; \Delta}\ \bot E_{\mathrm{I}}$$

The calculus Ncr

- $\vee$-elimination

  To prove $\Gamma \vdash F\Uparrow; \Delta$, with F prime, using $\vee$-elimination:

  - Non-deterministically select $H \in \Gamma$ such that $A \vee B \in \mathrm{Sf}^{+}(H)$
  - Start a $\downarrow$-expansion phase from the axiom sequent

    $$\Gamma_H ; H \vdash H\downarrow; F, \Delta ; \emptyset \qquad \Gamma_H = \Gamma \setminus \{H\}$$

    with the goal to extract $A \vee B$ from H

    $$\dfrac{}{\Gamma_H ; H \vdash H\downarrow; F, \Delta ; \emptyset}\,\mathrm{Id}$$
    $$\cdots$$
    $$\Gamma_H ; H \vdash A \vee B\downarrow; F, \Delta ; \Theta$$

  - Start an $\Uparrow$-expansion phase to prove $A, \Gamma_H, \Theta \vdash F\Uparrow; \Delta$
  - Start an $\Uparrow$-expansion phase to prove $B, \Gamma_H, \Theta \vdash F\Uparrow; \Delta$

  In the $\Uparrow$-expansion phases, H is replaced by the formulas in Θ.

$$\dfrac{\Gamma_H ; H \vdash A \vee B\downarrow; F, \Delta ; \Theta \quad A, \Gamma_H, \Theta \vdash F\Uparrow; \Delta \quad B, \Gamma_H, \Theta \vdash F\Uparrow; \Delta}{H, \Gamma \vdash F\Uparrow; \Delta}\,\vee E$$

The calculus **Ncr**

$$\frac{}{\Gamma\,;\,H \vdash\ H{\downarrow}\,;\,\Delta\,;}\ \mathrm{Id} \qquad \frac{\Gamma_H\,;\,H \vdash\ p{\downarrow}\,;\,p,\Delta\,;\,\Theta}{H,\Gamma\ \vdash\ p{\Uparrow}\,;\,\Delta}\ {\downarrow\!\Uparrow} \qquad \frac{\Gamma_H\,;\,H \vdash\ \bot{\downarrow}\,;\,F,\Delta\,;\,\Theta}{H,\Gamma\ \vdash\ F{\Uparrow}\,;\,\Delta}\ \bot E_{\mathrm{I}}$$

$$\frac{\Gamma_H\,;\,H \vdash\ p{\downarrow}\,;\,F,p,\Delta\,;\,\Theta}{H,\Gamma\ \vdash\ F{\Uparrow}\,;\,p,\Delta}\ \mathrm{R_p} \qquad \frac{\Gamma\ \vdash\ D{\Uparrow}\,;\,F,\Delta_D}{\Gamma\ \vdash\ F{\Uparrow}\,;\,D,\Delta}\ \mathrm{R_c} \quad D \notin \mathcal{V} \text{ and } p \neq \bot$$

$$\frac{\Gamma\ \vdash\ A{\Uparrow}\,;\,\Delta \qquad \Gamma\ \vdash\ B{\Uparrow}\,;\,\Delta}{\Gamma\ \vdash\ A \wedge B{\Uparrow}\,;\,\Delta}\ \wedge I \qquad \frac{\Gamma\,;\,H \vdash\ A_0 \wedge A_1{\downarrow}\,;\,\Delta\,;\,\Theta}{\Gamma\,;\,H \vdash\ A_k{\downarrow}\,;\,\Delta\,;\,A_{1-k},\Theta}\ \wedge E_k \quad k \in \{0,1\}$$

$$\frac{\Gamma\ \vdash\ A{\Uparrow}\,;\,B,\Delta}{\Gamma\ \vdash\ A \vee B{\Uparrow}\,;\,\Delta}\ \vee I$$

$$\frac{\Gamma_H\,;\,H \vdash\ A \vee B{\downarrow}\,;\,F,\Delta\,;\,\Theta \qquad A,\Gamma_H,\Theta\ \vdash\ F{\Uparrow}\,;\,\Delta \qquad B,\Gamma_H,\Theta\ \vdash\ F{\Uparrow}\,;\,\Delta}{H,\Gamma\ \vdash\ F{\Uparrow}\,;\,\Delta}\ \vee E$$

$$\frac{A,\Gamma\ \vdash\ B{\Uparrow}\,;\,\Delta}{\Gamma\ \vdash\ A \to B{\Uparrow}\,;\,\Delta}\ \to I \qquad \frac{\Gamma\,;\,H \vdash\ A \to B{\downarrow}\,;\,\Delta\,;\,\Theta \qquad \Gamma,\Theta\ \vdash\ A{\Uparrow}\,;\,\Delta}{\Gamma\,;\,H \vdash\ B{\downarrow}\,;\,\Delta\,;\,\Theta}\ \to E$$

$$p \in \mathcal{V},\ F \in \mathcal{V} \cup \{\bot\},\ \Gamma_H = \Gamma \setminus \{H\},\ \Delta_D = \Delta \setminus \{D\}$$

Properties of **Ncr**

- We can define a direct translation from **Ncr**-derivations into **Nc**, so that **Ncr** can be viewed as a *notational variant* of **Nc**.
- Differently from **Nc**, **Ncr** enjoys the strict subformula property.
- Branches of **Ncr** have finite length.
  Hence, the proof-search strategy is terminating (no loop-checking).
- No backtracking is needed (choices are non-deterministic)
- From the open proof-trees generated during a failed-proof search, we can extract a classical interpretation falsifying the initial sequent.

  This implies the completeness of **Ncr**.

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$\vdash \boldsymbol{p} \vee \neg \boldsymbol{p} \Uparrow ; \emptyset$$

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$\dfrac{\vdash\ p \Uparrow ; \neg p}{\vdash\ p \vee \neg p \Uparrow ; \emptyset}\ \vee I$$

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$\dfrac{\dfrac{\vdash \ \neg p \Uparrow ; \ p}{\vdash \ p \Uparrow ; \neg p}\ \mathrm{R_c}}{\vdash \ p \vee \neg p \Uparrow ; \ \emptyset}\ \vee I$$

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$
\dfrac{\dfrac{\dfrac{p \vdash \ \bot \Uparrow ;\ p}{\vdash \ \neg p \Uparrow ;\ p} \rightarrow I}{\vdash \ p \Uparrow ;\ \neg p} \mathrm{R_c}}{\vdash \ p \vee \neg p \Uparrow ;\ \emptyset} \vee I
$$

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\, p \vdash\ p{\downarrow}\,;\, \bot,\, p\,;\, \emptyset}\ \mathrm{Id}}{p \vdash\ \bot{\Uparrow}\,;\, p}\ \mathrm{R_p}}{\vdash\ \neg p{\Uparrow}\,;\, p}\ \to I}{\vdash\ p{\Uparrow}\,;\, \neg p}\ \mathrm{R_c}}{\vdash\ p \vee \neg p{\Uparrow}\,;\, \emptyset}\ \vee I
$$

Example 1

Let us prove $p \vee \neg p$ in **Ncr**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\, p \vdash\ p{\downarrow}\,;\,\bot,\, p\,;\,\emptyset}\ \mathrm{Id}}{p\ \vdash\ \bot{\Uparrow}\,;\, p}\ \mathrm{R_p}}{\vdash\ \neg p{\Uparrow}\,;\, p}\ \to I}{\vdash\ p{\Uparrow}\,;\,\neg p}\ \mathrm{R_c}}{\vdash\ p \vee \neg p{\Uparrow}\,;\,\emptyset}\ \vee I
$$

We have only one $\Uparrow$-expansion phase followed by a trivial $\downarrow$-expansion phase

Similar to the sequent derivation

$$
\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\ \mathsf{Ax}}{\Rightarrow p, \neg p}\ R\to}{\Rightarrow p \vee \neg p}\ R\vee
$$

Example 2

Let us prove $p \wedge (p \to q) \to q$ in **Ncr**

Example 2

Let us prove $p \wedge (p \rightarrow q) \rightarrow q$ in **Ncr**

$$\dfrac{}{(0) \quad \vdash p \wedge (p \rightarrow q) \rightarrow q \Uparrow ; \emptyset} \rightarrow I$$

Example 2

Let us prove $p \wedge (p \to q) \to q$ in **Ncr**

$$\dfrac{(1) \quad p \wedge (p \to q) \vdash q \Uparrow ; \emptyset}{(0) \quad \vdash p \wedge (p \to q) \to q \Uparrow ; \emptyset} \to I$$

Example 2

Let us prove $p \wedge (p \to q) \ \to \ q$ in **Ncr**

$$\dfrac{}{(2) \quad \emptyset\,;\, p \wedge (p \to q) \vdash p \wedge (p \to q){\downarrow}\,;\, q\,;\, \emptyset}\ \mathrm{Id}$$

$$\dfrac{(1) \quad p \wedge (p \to q) \vdash q{\Uparrow}\,;\, \emptyset}{(0) \quad \vdash p \wedge (p \to q) \ \to \ q{\Uparrow}\,;\, \emptyset} \to I$$

Example 2

Let us prove $p \wedge (p \to q) \to q$ in **Ncr**

$$\dfrac{\dfrac{}{(2)\quad \emptyset\,;\, p \wedge (p \to q) \vdash p \wedge (p \to q){\downarrow}\,;\, q\,;\, \emptyset}\ \mathrm{Id}}{(3)\quad \emptyset\,;\, p \wedge (p \to q) \vdash p \to q{\downarrow}\,;}\ \wedge E_1$$

$$\dfrac{(1)\quad p \wedge (p \to q) \vdash q{\Uparrow}\,;\, \emptyset}{(0)\quad \vdash p \wedge (p \to q) \to q{\Uparrow}\,;\, \emptyset} \to I$$

Example 2

Let us prove $p \wedge (p \to q) \to q$ in **Ncr**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(2)\quad \emptyset ; p \wedge (p \to q) \vdash p \wedge (p \to q){\downarrow} ; q ; \emptyset}\ \mathrm{Id}}{(3)\quad \emptyset ; p \wedge (p \to q) \vdash p \to q{\downarrow} ; q ; p}\ \wedge E_1 \qquad (5)\quad p \vdash p{\Uparrow} ; q}{(4)\quad \emptyset ; p \wedge (p \to q) \vdash q{\downarrow} ; q ; p}\ \to E}{(1)\quad p \wedge (p \to q) \vdash q{\Uparrow} ; \emptyset}\ \downarrow\!\Uparrow}{(0)\quad \vdash p \wedge (p \to q) \to q{\Uparrow} ; \emptyset}\ \to I
$$

Example 2

Let us prove $p \wedge (p \to q) \to q$ in **Ncr**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(2)\quad \emptyset;\, p\wedge(p\to q) \vdash p\wedge(p\to q)\downarrow;\, q;\, \emptyset}\,\text{Id}}{(3)\quad \emptyset;\, p\wedge(p\to q) \vdash p\to q\downarrow;\, q;\, p}\,\wedge E_1 \qquad \dfrac{\dfrac{}{(6)\quad \emptyset;\, p \vdash p\downarrow;\, q, p;\, \emptyset}\,\text{Id}}{(5)\quad p \vdash p\Uparrow;\, q}\,\downarrow\Uparrow}{(4)\quad \emptyset;\, p\wedge(p\to q) \vdash q\downarrow;\, q;\, p}\,\to E}{(1)\quad p\wedge(p\to q) \vdash q\Uparrow;\, \emptyset}\,\downarrow\Uparrow}{(0)\quad \vdash p\wedge(p\to q) \to q\Uparrow;\, \emptyset}\,\to I$$

Example 2

Let us prove $p \wedge (p \to q) \;\to\; q$ in **Ncr**

$$
\dfrac{
  \dfrac{
    \dfrac{}{(2)\quad \emptyset\,;\, p \wedge (p \to q) \vdash \color{blue}{p \wedge (p \to q)}\!\downarrow ;\, q\,;\, \emptyset}\ \mathrm{Id}
  }{(3)\quad \emptyset\,;\, p \wedge (p \to q) \vdash \color{blue}{p \to q}\!\downarrow ;\, q\,;\, \color{blue}{p}}\ \color{blue}{\wedge E_1}
  \qquad
  \dfrac{
    \dfrac{}{(6)\quad \emptyset\,;\, p \vdash p\!\downarrow ;\, q,\, p\,;\, \emptyset}\ \mathrm{Id}
  }{(5)\quad p \vdash p\Uparrow ;\, q}\ \downarrow\!\Uparrow
}{
  \dfrac{
    \dfrac{(4)\quad \emptyset\,;\, p \wedge (p \to q) \vdash q\!\downarrow ;\, q\,;\, p}{(1)\quad p \wedge (p \to q) \vdash q\Uparrow ;\, \emptyset}\ \downarrow\!\Uparrow
  }{(0)\quad \vdash p \wedge (p \to q) \;\to\; q\Uparrow ;\, \emptyset}\ \to I
}\ \to E
$$

Only one $\wedge$-elimination, as in sequent calculus!

Example 2

Let us prove $p \land (p \to q) \to q$ in **Ncr**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(2)\quad \emptyset\,;\, p \land (p \to q) \vdash p \land (p \to q)\!\downarrow;\, q\,;\, \emptyset}\,\mathrm{Id}}{(3)\quad \emptyset\,;\, p \land (p \to q) \vdash p \to q\!\downarrow;\, q\,;\, p}\,\land E_1 \qquad \dfrac{\dfrac{}{(6)\quad \emptyset\,;\, p \vdash p\!\downarrow;\, q,\, p\,;\, \emptyset}\,\mathrm{Id}}{(5)\quad p \vdash p\!\Uparrow;\, q}\,\downharpoonleft\!\Uparrow}{(4)\quad \emptyset\,;\, p \land (p \to q) \vdash q\!\downarrow;\, q\,;\, p}\,\to E}{(1)\quad p \land (p \to q) \vdash q\!\Uparrow;\, \emptyset}\,\downharpoonleft\!\Uparrow}{(0)\quad \vdash p \land (p \to q) \to q\!\Uparrow;\, \emptyset}\,\to I$$

Only one $\land$-elimination, as in sequent calculus!

$$\dfrac{\dfrac{\dfrac{}{p, p \to q \Rightarrow p}\,\mathrm{Ax} \qquad \dfrac{}{p, q \Rightarrow q}\,\mathrm{Ax}}{\dfrac{p, p \to q \Rightarrow q}{p \land (p \to q) \Rightarrow q}\,L\land}\,L \to}{\Rightarrow p \land (p \to q) \to q}\,R \to$$

Related work and Conclusion

- We have presented a procedure to build derivations in **Ncr** not requiring backtracking nor loop-checking.
  The strategy alternates $\Uparrow$ and $\Downarrow$-expansion phases.
  - Each phase focuses on a formula and eagerly decomposes it.
  - When in $\Uparrow$-expansion we get a prime formula, we can:
    (a) continue $\Uparrow$-expansion, restarting from a non-prime formula Or
    (b) non-deterministically select a head formula to start a new $\Downarrow$-expansion phase

  There is some high-level analogy with focused calculi, nevertheless **Ncr** cannot be classified as such (no polarization of connectives and atoms).
- **Ncr**-derivations have a direct translation into derivations of Gentzen natural deduction calculus in normal form.
- If we restrict ourselves to the $\{\rightarrow, \bot\}$-fragment of the language, the procedure behaves like the goal-oriented proof-search strategy of [Gabbay&Olivetti,2000]

Related work and Conclusion

- The idea of performing proof-search in natural deduction calculi applying I-rules bottom-up and E-rules top-down, so to build derivations in normal form, dates back to Sieg work.

  The naïve proof-search strategy is highly inefficient, due to the huge number of backtrack points; moreover, to guarantee termination, one has to check that a configuration does not occur twice along a branch.

Related work and Conclusion

- Natural deduction-like calculi have also been employed to implement first-order theorem provers, see e.g.

A. Bolotov, V. Bocharov, A. Gorchakov, and V. Shangin. Automated first order natural deduction. IICAI, *2005.*

A. Indrzejczak. Natural Deduction, Hybrid Systems and Modal Logics, *of* Trends in Logic, *2010*

D. Pastre. Strong and weak points of the MUSCADET theorem prover - examples from CASC-JC. AI Commun., *2002.*

  In these systems, the goal is to implement reasoning in first-order logic in natural deduction style (introduction and elimination of assumptions).
  Proof-search requires the inspection of the whole database of available assumptions.

- Working implementation:
  `http://www.dista.uninsubria.it/~ferram/`.