A terminating evaluation-driven variant of **G3i**

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Motivations

G3i

Single-succedent sequent calculus for intuitionistic propositional logic **IPL** where weakening and contraction are "absorbed" into the rules.

$$\frac{}{\bot, \Gamma \Rightarrow H}\ \bot L \qquad \frac{}{H, \Gamma \Rightarrow H}\ \mathrm{Id}$$

$$\frac{A, B, \Gamma \Rightarrow H}{A \wedge B, \Gamma \Rightarrow H}\ \wedge L \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R$$

$$\frac{A, \Gamma \Rightarrow H \quad B, \Gamma \Rightarrow H}{A \vee B, \Gamma \Rightarrow H}\ \vee L \qquad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_0 \vee A_1}\ \vee R_j$$

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow H}{A \to B, \Gamma \Rightarrow H}\ \to L \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to R$$

Troelstra & Schwichtenberg, Basic Proof Theory, 1996

Motivations

It is well-known that **G3i** is not suited for backward proof-search.

The problem arises from the rule

$$\dfrac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow H}{A \to B, \Gamma \Rightarrow H} \to L$$

In bottom-up proof-search, this rules might generate non-terminating branches.

$$\dfrac{\dfrac{\dfrac{\vdots}{A \to B, \Gamma \Rightarrow A} \qquad B, \Gamma \Rightarrow A}{A \to B, \Gamma \Rightarrow A} \to L \qquad B, \Gamma \Rightarrow H}{A \to B, \Gamma \Rightarrow H} \to L$$

Motivations

To narrow the search space and get a terminating proof-search procedure, one has to implement some auxiliary mechanism.

- Loop-checking

  Whenever the "same" sequent occurs twice along a branch of the proof under construction, the search is cut.

- Histories (an efficient implementation of loop-checking).

  In the construction of a branch, some of the right formulas are stored in the history.
  Some rule applications require a local check to the history set.

A. Heuerding et al., *Efficient loop-check for backward proof search in some non-classical propositional logics*, Tableaux 96.

J. M. Howe., *Two loop detection mechanisms: A comparison*, Tableaux 97.

D.M. Gabbay and N. Olivetti, *Goal-Directed Proof Theory*, 2000.

Our approach

We show that terminating proof-search for **G3i** can be performed only exploiting the information contained in the sequent to be proved.

- History based approach
  Auxiliary sets of formulas are introduced.
- Our approach
  Termination is controlled by an evaluation relation defined on sequents.

Labelled sequents

The proof-search strategy alternates two phases (unblocked and blocked).

The strategy is embedded in the calculus by annotating sequents with a label $l \in \{\mathrm{u}, \mathrm{b}\}$.

- Unblocked sequent (u-sequent)

  $$\Gamma \stackrel{\mathrm{u}}{\Rightarrow} H \qquad \Gamma \text{ is a set of formulas}$$

  Any rule can be backward applied (like *ordinary* sequents)

- Blocked sequent (b-sequent)

  $$\Gamma \stackrel{\mathrm{b}}{\Rightarrow} H$$

  Only right rules can be applied and left context is blocked (see *right-focused* sequents)

Proof-search starts from an u-sequent (u-phase).

Towards **Gbu**

G3i $+$ Labels $\mathrm{b,u}$ $+$ Evaluation relation $\Longrightarrow$ **Gbu**

- Labels
  Mark the current phase
- Evaluation relation
  - Used in the definition of the rules for right implication.
  - Crucial to get termination.

Overview of the calculus **Gbu**

- Axiom rules

$$\frac{}{\bot, \Gamma \xRightarrow{l} H}\ \bot L \qquad \frac{}{H, \Gamma \xRightarrow{l} H}\ \mathrm{Id} \qquad l \in \{\mathrm{b}, \mathrm{u}\}$$

  Axiom rules of **G3i** + labels

- Rules for $\wedge$, $\vee$ and left $\rightarrow$

  Rules of **G3i** + labels

- Right $\rightarrow$

  Two labelled variants of the rule $\rightarrow R$ of **G3i**.
  Labels are determined by the evaluation relation

Rules preserving the u-phase

Backward proof-search starts from an u-sequent.

- Left and right conjunction

$$\frac{A, B, \Gamma \overset{u}{\Rightarrow} H}{A \wedge B, \Gamma \overset{u}{\Rightarrow} H}\ \wedge L \qquad\qquad \frac{\Gamma \overset{u}{\Rightarrow} A \qquad \Gamma \overset{u}{\Rightarrow} B}{\Gamma \overset{u}{\Rightarrow} A \wedge B}\ \wedge R$$

- Left disjunction

$$\frac{A, \Gamma \overset{u}{\Rightarrow} H \qquad B, \Gamma \overset{u}{\Rightarrow} H}{A \vee B, \Gamma \overset{u}{\Rightarrow} H}\ \vee L$$

Note

In left-rules, the main formula does not belong to Γ.

Switch from u-phase to b-phase

- Right disjunction

$$\frac{\Gamma \overset{b}{\Rightarrow} A_j}{\Gamma \overset{u}{\Rightarrow} A_0 \vee A_1} \vee R_j \quad j \in \{0, 1\}$$

- Left implication

$$\frac{A \to B, \Gamma \overset{b}{\Rightarrow} A \qquad B, \Gamma \overset{u}{\Rightarrow} H}{A \to B, \Gamma \overset{u}{\Rightarrow} H} \to L$$

Rules preserving the b-phase

In a b-phase only right rules can be applied (right focus).

- Right conjunction

$$\frac{\Gamma \overset{b}{\Rightarrow} A \qquad \Gamma \overset{b}{\Rightarrow} B}{\Gamma \overset{b}{\Rightarrow} A \wedge B} \wedge R$$

- Right disjunction

$$\frac{\Gamma \overset{b}{\Rightarrow} A_j}{\Gamma \overset{b}{\Rightarrow} A_0 \vee A_1} \vee R_j \qquad j \in \{0, 1\}$$

Evaluation relations

An evaluation relation $\vdash_{\mathcal{E}}$ is a relation between a set of formulas Γ and a formula A.

Intuitively

$$\Gamma \vdash_{\mathcal{E}} A$$

means

the truth of A is entailed by Γ

The calculus **Gbu** does not rely on a specific evaluation relation $\vdash_{\mathcal{E}}$.
We can use *any* $\vdash_{\mathcal{E}}$ satisfying the next properties

Properties of $\vdash_{\mathcal{E}}$

1. $\Gamma \vdash_{\mathcal{E}} A$ iff $\Gamma \cap \mathrm{Subf}(A) \vdash_{\mathcal{E}} A$.

   To evaluate A in Γ, only the formulas of Γ which are subformulas of A are relevant.

2. $A, \Gamma \vdash_{\mathcal{E}} A$.
3. $\Gamma \vdash_{\mathcal{E}} A$ and $\Gamma \vdash_{\mathcal{E}} B$ implies $\Gamma \vdash_{\mathcal{E}} A \wedge B$.
4. $\Gamma \vdash_{\mathcal{E}} A$ or $\Gamma \vdash_{\mathcal{E}} B$ implies $\Gamma \vdash_{\mathcal{E}} A \vee B$.
5. $\Gamma \vdash_{\mathcal{E}} B$ implies $\Gamma \vdash_{\mathcal{E}} A \rightarrow B$.
6. Semantical condition
   Let $\mathcal{K}$ be a Kripke model and α a world of $\mathcal{K}$.
   If $\mathcal{K}, \alpha \Vdash \Gamma$ (all the formulas of Γ are forced in α)
   and $\Gamma \vdash_{\mathcal{E}} A$
   then $\mathcal{K}, \alpha \Vdash A$.

The evaluation relation $\vdash_{\tilde{\mathcal{E}}}$

In our implementation of **Gbu**, we use the evaluation relation $\vdash_{\tilde{\mathcal{E}}}$

To check if $\Gamma \vdash_{\tilde{\mathcal{E}}} A$:

(i) Replace every $B \in \mathrm{Subf}(A) \cap \Gamma$ by $\top$

(ii) Apply the following boolean simplifications inside formulas:

$$K \wedge \top \rightsquigarrow K \qquad K \wedge \bot \rightsquigarrow \bot \qquad K \vee \top \rightsquigarrow \top \qquad K \vee \bot \rightsquigarrow K$$

$$K \to \top \rightsquigarrow \top \qquad \top \to K \rightsquigarrow K \qquad \bot \to K \rightsquigarrow \top$$

$\Gamma \vdash_{\tilde{\mathcal{E}}} A$ iff at the end of steps (i)–(ii) we get $\top$.

Example

Let

$$\Gamma \;=\; \{\, A,\; B \,\}$$

Examples of formulas F such that

$$\Gamma \;\vdash_{\tilde{\mathcal{E}}}\; F$$

F	Replace	Simplify
$(A \wedge B) \vee C$	$(\top \wedge \top) \vee C$	$\rightsquigarrow\ \top$
$C \rightarrow (A \vee D)$	$C \rightarrow \top \vee D$	$\rightsquigarrow\ \top$

The evaluation relation $\vdash_{\tilde{\mathcal{E}}}$

Formal definition of $\vdash_{\tilde{\mathcal{E}}}$

$$\mathcal{R}(A,\Gamma) = \begin{cases} \top & A \in \Gamma \\ A & \text{if } A \notin \Gamma \text{ and } A \text{ atomic} \\ & \text{(namely, } A \in \mathcal{V} \cup \{\bot, \top\}) \\ \mathcal{B}\left(\mathcal{R}(A_0,\Gamma) \cdot \mathcal{R}(A_1,\Gamma)\right) & \text{if } A \notin \Gamma \text{ and } A = A_0 \cdot A_1 \\ & \text{with } \cdot \in \{\wedge, \vee, \rightarrow\} \end{cases}$$

$\mathcal{B}(A)$: formula obtained by applying boolean simplifications to A.

$$\Gamma \vdash_{\tilde{\mathcal{E}}} A \quad \text{IFF} \quad \mathcal{R}(A,\Gamma) = \top$$

Proposition

$\vdash_{\tilde{\mathcal{E}}}$ is an evaluation relation

Rules for right-implication

Backward application of right-implication to

$$\Gamma \overset{l}{\Rightarrow} A \to B \qquad l \in \{\mathrm{b}, \mathrm{u}\}$$

$$\Gamma \vdash_{\mathcal{E}} A \; ?$$

- If $\Gamma \vdash_{\mathcal{E}} A$:

$$\frac{\Gamma \overset{l}{\Rightarrow} B}{\Gamma \overset{l}{\Rightarrow} A \to B} \to R_1$$

  The phase $l \in \{\mathrm{b}, \mathrm{u}\}$ does not change.
  A is not added to the left context (difference from **G3i**)

- If $\Gamma \nvdash_{\mathcal{E}} A$:

$$\frac{A, \Gamma \overset{u}{\Rightarrow} B}{\Gamma \overset{l}{\Rightarrow} A \to B} \to R_2$$

  This is the only rule that *unblocks* a b-phase.

The calculus **Gbu**

$$\frac{}{\bot, \Gamma \overset{l}{\Rightarrow} H} \;\bot L \qquad\qquad \frac{}{H, \Gamma \overset{l}{\Rightarrow} H} \;\mathrm{Id}$$

$$\frac{A, B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H}{A \wedge B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H} \;\wedge L \qquad\qquad \frac{\Gamma \overset{l}{\Rightarrow} A \quad \Gamma \overset{l}{\Rightarrow} B}{\Gamma \overset{l}{\Rightarrow} A \wedge B} \;\wedge R$$

$$\frac{A, \Gamma \overset{\mathrm{u}}{\Rightarrow} H \quad B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H}{A \vee B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H} \;\vee L \qquad\qquad \frac{\Gamma \overset{\mathrm{b}}{\Rightarrow} A_j}{\Gamma \overset{l}{\Rightarrow} A_0 \vee A_1} \;\vee R_j \qquad j \in \{0, 1\}$$

$$\frac{A \to B, \Gamma \overset{\mathrm{b}}{\Rightarrow} A \quad B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H}{A \to B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H} \;\to L \qquad \frac{\Gamma \overset{l}{\Rightarrow} B}{\Gamma \overset{l}{\Rightarrow} A \to B} \;\to R_1 \qquad \frac{A, \Gamma \overset{\mathrm{u}}{\Rightarrow} B}{\Gamma \overset{l}{\Rightarrow} A \to B} \;\to R_2$$

if $\Gamma \vdash_{\mathcal{E}} A$ if $\Gamma \nvdash_{\mathcal{E}} A$

The calculus **Gbu**

Erasing the labels and weakening rule $\to R_1$, we get **G3i**.

$$\dfrac{}{\bot, \Gamma \Rightarrow H}\ \bot L \qquad \dfrac{}{H, \Gamma \Rightarrow H}\ \mathrm{Id}$$

$$\dfrac{A, B, \Gamma \Rightarrow H}{A \wedge B, \Gamma \Rightarrow H}\ \wedge L \qquad \dfrac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R$$

$$\dfrac{A, \Gamma \Rightarrow H \quad B, \Gamma \Rightarrow H}{A \vee B, \Gamma \Rightarrow H}\ \vee L \qquad \dfrac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_0 \vee A_1}\ \vee R_j \quad j \in \{0, 1\}$$

$$\dfrac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow H}{A \to B, \Gamma \Rightarrow H}\ \to L \qquad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to R_1 \qquad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to R_2$$

if $\Gamma \vdash_{\mathcal{E}} A$ (for $\to R_1$) if $\Gamma \nvdash_{\mathcal{E}} A$ (for $\to R_2$)

Properties of **Gbu**-trees

Structure of a branch of a **Gbu**-tree with root $\Gamma \overset{\mathrm{u}}{\Rightarrow} H$

u-phase

$A, \Gamma_2 \overset{u}{\Rightarrow} B$

Phase switch (rule $\to R_2$)

$\Gamma_2 \overset{b}{\Rightarrow} A \to B$ $\quad$ $\Gamma_2 \nvdash_{\varepsilon} A$

b-phase

$\Gamma_2 \overset{b}{\Rightarrow} H_2$

Phase switch (rule $\to L$ or $\vee R$)

$\Gamma_1 \overset{u}{\Rightarrow} H_1$

u-phase

$\Gamma \overset{u}{\Rightarrow} H$

Properties of **Gbu**-trees

Let $\mathcal{B}$ be a branch with root sequent σ.
Let $|\sigma|$ be the size of σ (= number of symbols occurring in σ).

- The construction of $\mathcal{B}$ ends when:
  - (i) an axiom rule of **Gbu** is applied OR
  - (ii) no rule of **Gbu** can be applied.

- **Gbu** has the subformula property.

  Hence, for every formula A occurring in $\mathcal{B}$, $A \in \mathrm{Subf}(\sigma)$.

Properties of **Gbu**-trees

Let $\mathcal{B}$ be a branch with root sequent σ.

- Along $\mathcal{B}$, we have at most $|\sigma|$ applications of $\rightarrow R_2$.

  Idea

  When in the bottom up construction of $\mathcal{B}$ the rule

$$\frac{A, \Gamma \overset{\mathrm{u}}{\Rightarrow} B}{\Gamma \overset{\mathrm{b}}{\Rightarrow} A \rightarrow B} \rightarrow R_2$$

  is applied, we have

$$\Gamma \nvdash_{\mathcal{E}} A \qquad A \text{ actually adds } \textit{new} \text{ information to } \Gamma$$

  By properties of $\vdash_{\mathcal{E}}$, it follows that:

  for every $\Gamma' \overset{l}{\Rightarrow} H'$ in $\mathcal{B}$ below $\Gamma \overset{\mathrm{b}}{\Rightarrow} A \rightarrow B$, $A \notin \Gamma'$.

  Hence, we cannot apply twice $\rightarrow R_2$ to the same formula $A \rightarrow B$.

  Since $A \rightarrow B \in \mathrm{Subf}(\sigma)$, there are at most $|\sigma|$ applications of $\rightarrow R_2$.

Properties of **Gbu**-trees

$A_3, \Gamma_3 \overset{\mathrm{u}}{\Rightarrow} B_3$

$\Gamma_3 \overset{\mathrm{b}}{\Rightarrow} A_3 \to B_3$ $\quad \to R_2 \qquad \Gamma_3 \nvdash_{\mathcal{E}} A_3 \qquad \Gamma_3 \vdash_{\mathcal{E}} A_1$

(u + b)-phase (3)

$A_2, \Gamma_2 \overset{\mathrm{u}}{\Rightarrow} B_2$

$\Gamma_2 \overset{\mathrm{b}}{\Rightarrow} A_2 \to B_2$ $\quad \to R_2 \qquad \Gamma_2 \nvdash_{\mathcal{E}} A_2 \qquad \Gamma_2 \vdash_{\mathcal{E}} A_1$

(u + b)-phase (2)

$A_1, \Gamma_1 \overset{\mathrm{u}}{\Rightarrow} B_1$ $\qquad A_1, \Gamma_1 \vdash_{\mathcal{E}} A_1$

$\Gamma_1 \overset{\mathrm{b}}{\Rightarrow} A_1 \to B_1$ $\quad \to R_2 \qquad \Gamma_1 \nvdash_{\mathcal{E}} A_1$

(u + b)-phase (1)

By properties of $\vdash_{\mathcal{E}}$, it follows that:

$$A_1, \Gamma_1 \vdash_{\mathcal{E}} A_1 \qquad \Gamma_2 \vdash_{\mathcal{E}} A_1 \qquad \Gamma_3 \vdash_{\mathcal{E}} A_1$$

Hence, the main formulas $A_j \to B_j$ of $\to R_2$ are pairwise disjoint.

Properties of **Gbu**-trees

Let $\mathcal{B}$ be a branch with root sequent σ.

- In $\mathcal{B}$ we have at most:

$$\begin{array}{rl} |\sigma| & \text{switches from b to u } (\to R_2 \text{ applications}) \\ |\sigma| + 1 & \text{switches from u to b} \end{array}$$

- The size of sequents can only increase by an application of rule $\to L$ (switch from u to b)

$$\frac{A \to B, \Gamma \overset{\mathrm{b}}{\Rightarrow} A}{A \to B, \Gamma \overset{\mathrm{u}}{\Rightarrow} H} \to L$$

The length of $\mathcal{B}$ is at most $|\sigma|^2$ (optimal bound).

Buss and R. Iemhoff. *The depth of intuitionistic cut free proofs*. 2003

Soundness and Completeness of **Gbu**

$$\Gamma \Rightarrow H \text{ is provable in } \mathbf{G3i} \quad \Longleftrightarrow \quad \Gamma \overset{u}{\Rightarrow} H \text{ is provable in } \mathbf{Gbu}$$

$$A \in \mathbf{IPL} \quad \Longleftrightarrow \quad \overset{u}{\Rightarrow} A \text{ is provable in } \mathbf{Gbu}$$

- Soundness ($\Longleftarrow$)

  Trivial

$$\begin{array}{c}\Pi\\ \Gamma \overset{u}{\Rightarrow} H\end{array} \text{ in } \mathbf{Gbu} \quad \mapsto \quad \begin{array}{c}\Pi*\\ \Gamma \Rightarrow H\end{array} \text{ in } \mathbf{G3i}$$

- Completeness ($\Longrightarrow$)

  Tricky

$$\begin{array}{c}\Pi\\ \Gamma \Rightarrow H\end{array} \text{ in } \mathbf{G3i} \quad \overset{?}{\mapsto} \quad \begin{array}{c}\Pi*\\ \Gamma \overset{u}{\Rightarrow} H\end{array} \text{ in } \mathbf{Gbu}$$

Is there a translation from **G3i** into **Gbu** ?

Completeness of **Gbu**

We prove completeness using Kripke semantics along the lines of

L. Pinto and R. Dyckhoff. *Loop-free construction of counter-models for intuitionistic propositional logic*. 1995

M. Ferrari, C. Fiorentini, and G. Fiorino. *Contraction-free linear depth sequent calculi for intuitionistic propositional logic with the subformula property and minimal depth counter-models*, JAR, 2013

- We introduce a refutation calculus **Rbu** for asserting intuitionistic unprovability (a dual calculus of **Gbu**).
- From an **Rbu**-derivation of $\Gamma \stackrel{u}{\Rightarrow} H$ we can extract a countermodel $\mathcal{K}$ of $\Gamma \Rightarrow H$, namely:
  - $\mathcal{K}$ is a Kripke model such that, at its root, all formulas in Γ are forced and H is not forced.
- If the search for a **Gbu**-derivation of $\Gamma \stackrel{u}{\Rightarrow} H$ fails, then we can build an **Rbu**-derivation of $\Gamma \stackrel{u}{\Rightarrow} H$.

The proof-search procedure

We provide a terminating proof-search procedure based on backward application of rules of **Gbu**.

Input: $\Gamma \overset{u}{\Rightarrow} H$

Output:

(i) A **Gbu**-derivation of $\Gamma \overset{u}{\Rightarrow} A$ OR

(ii) A **Rbu**-derivation of $\Gamma \overset{u}{\Rightarrow} A$

(i) can be immediately translated to a **G3i**-derivation of $\Gamma \Rightarrow A$

(ii) yields a countermodel for $\Gamma \Rightarrow A$.

A proof-search example (1)

Let us search for a derivation for the formula

$$W \;=\; ((((p \to q) \to p) \to p) \to q) \to q \quad \text{(Weak Pierce Law)}$$

Backward proof-search starts with the unblocked sequent

$$\stackrel{\mathrm{u}}{\Rightarrow} W$$

We can only apply $\to R_2$ with main formula W.

A proof-search example (2)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\frac{A \overset{\mathrm{u}}{\Rightarrow} q \;_{2}}{\overset{\mathrm{u}}{\Rightarrow} W \;_{1}} \to R_2$$

Sequent 2

We can only apply $\to L$ with main formula A.

A proof-search example (3)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\dfrac{A \overset{\mathrm{b}}{\Rightarrow} B \to p \;\; {}_3 \qquad \dfrac{}{q \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_4}\,\mathrm{Id}}{\dfrac{A \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_2}{\overset{\mathrm{u}}{\Rightarrow} W \;\; {}_1}\to R_2}\to L$$

Sequent 3 is blocked.

We can only apply $\to R_2$ with main formula $B \to p$

A proof-search example (4)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\dfrac{\dfrac{\dfrac{B, A \overset{\mathrm{u}}{\Rightarrow} p \;_5}{A \overset{\mathrm{b}}{\Rightarrow} B \to p \;_3} \to R_2 \qquad \dfrac{}{q \overset{\mathrm{u}}{\Rightarrow} q \;_4}\,\mathrm{Id}}{A \overset{\mathrm{u}}{\Rightarrow} q \;_2} \to L}{\overset{\mathrm{u}}{\Rightarrow} W \;_1} \to R_2$$

Sequent 5

We can apply $\to L$ with main formula B or A (backtrack point).

We choose A.

A proof-search example (5)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\dfrac{\dfrac{\dfrac{B, A \overset{\mathrm{b}}{\Rightarrow} p \to q \;_{6} \qquad \dfrac{}{p, A \overset{\mathrm{u}}{\Rightarrow} p \;_{7}}\ \mathrm{Id}}{B, A \overset{\mathrm{u}}{\Rightarrow} p \;_{5}}\ {\to}L}{A \overset{\mathrm{b}}{\Rightarrow} B \to p \;_{3}}\ {\to}R_2 \qquad \dfrac{}{q \overset{\mathrm{u}}{\Rightarrow} q \;_{4}}\ \mathrm{Id}}{\dfrac{A \overset{\mathrm{u}}{\Rightarrow} q \;_{2}}{\overset{\mathrm{u}}{\Rightarrow} W \;_{1}}\ {\to}R_2}\ {\to}L$$

Sequent 6 is blocked

We can only apply $\to R_2$ with main formula $p \to q$

A proof-search example (6)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{p, B, A \stackrel{u}{\Rightarrow} q \;\; 8}{B, A \stackrel{b}{\Rightarrow} p \to q \;\; 6} \to R_2
      \qquad
      \dfrac{}{p, A \stackrel{u}{\Rightarrow} p \;\; 7}\,\mathrm{Id}
    }{B, A \stackrel{u}{\Rightarrow} p \;\; 5} \to L
  }{A \stackrel{b}{\Rightarrow} B \to p \;\; 3} \to R_2
  \qquad
  \dfrac{}{q \stackrel{u}{\Rightarrow} q \;\; 4}\,\mathrm{Id}
}{
  \dfrac{A \stackrel{u}{\Rightarrow} q \;\; 2}{\stackrel{u}{\Rightarrow} W \;\; 1} \to R_2
} \to L
$$

Sequent 8: we can apply $\to L$ with main formula B or A.

We choose A.

A proof-search example (7)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\dfrac{p, B, A \overset{\mathrm{b}}{\Rightarrow} B \to p \;\; {}_9 \qquad \dfrac{}{q, p, B \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_{10}}\,\mathsf{Id}}{p, B, A \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_8}\,\to L$$

$$\dfrac{p, B, A \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_8}{B, A \overset{\mathrm{b}}{\Rightarrow} p \to q \;\; {}_6}\,\to R_2$$

$$\dfrac{B, A \overset{\mathrm{b}}{\Rightarrow} p \to q \;\; {}_6 \qquad \dfrac{}{p, A \overset{\mathrm{u}}{\Rightarrow} p \;\; {}_7}\,\mathsf{Id}}{B, A \overset{\mathrm{u}}{\Rightarrow} p \;\; {}_5}\,\to L$$

$$\dfrac{B, A \overset{\mathrm{u}}{\Rightarrow} p \;\; {}_5}{A \overset{\mathrm{b}}{\Rightarrow} B \to p \;\; {}_3}\,\to R_2$$

$$\dfrac{A \overset{\mathrm{b}}{\Rightarrow} B \to p \;\; {}_3 \qquad \dfrac{}{q \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_4}}{A \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_2}$$

$$\dfrac{A \overset{\mathrm{u}}{\Rightarrow} q \;\; {}_2}{\overset{\mathrm{u}}{\Rightarrow} W \;\; {}_1}\,\to R_2$$

Sequent 9 is blocked and

$$p, B, A \vdash_{\tilde{\mathcal{E}}} B$$

We have to apply $\to R_1$ with main formula $B \to p$

A proof-search example (8)

$$W \;=\; A \rightarrow q \qquad A \;=\; (B \rightarrow p) \rightarrow q \qquad B \;=\; (p \rightarrow q) \rightarrow p$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{}{p, B, A \overset{\text{b}}{\Rightarrow} p \;\; {}_{11}}\,\text{Id}
            }{p, B, A \overset{\text{b}}{\Rightarrow} {\color{red}B \rightarrow p} \;\; {}_{9}}\,{\color{red}\rightarrow R_1}
            \qquad
            \dfrac{}{q, p, B \overset{\text{u}}{\Rightarrow} q \;\; {}_{10}}\,\text{Id}
          }{p, B, A \overset{\text{u}}{\Rightarrow} q \;\; {}_{8}}\,\rightarrow L
        }{B, A \overset{\text{b}}{\Rightarrow} p \rightarrow q \;\; {}_{6}}\,\rightarrow R_2
        \qquad
        \dfrac{}{p, A \overset{\text{u}}{\Rightarrow} p \;\; {}_{7}}\,\text{Id}
      }{B, A \overset{\text{u}}{\Rightarrow} p \;\; {}_{5}}\,\rightarrow L
    }{A \overset{\text{b}}{\Rightarrow} B \rightarrow p \;\; {}_{3}}\,\rightarrow R_2
    \qquad
    \dfrac{}{q \overset{\text{u}}{\Rightarrow} q \;\; {}_{4}}
  }{A \overset{\text{u}}{\Rightarrow} q \;\; {}_{2}}
}{\overset{\text{u}}{\Rightarrow} W \;\; {}_{1}}\,\rightarrow R_2
$$

We have built a **Gbu**-derivation of $\overset{\text{u}}{\Rightarrow} W$.

Erasing the labels ...

A proof-search example (9)

$$W \;=\; A \to q \qquad A \;=\; (B \to p) \to q \qquad B \;=\; (p \to q) \to p$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, B, A \Rightarrow p \ _{11}}\,\text{Id}}{p, B, A \Rightarrow B \to p \ _{9}}\to R \quad \dfrac{}{q, p, B \Rightarrow q \ _{10}}\,\text{Id}}{p, B, A \Rightarrow q \ _{8}}\to L}{B, A \Rightarrow p \to q \ _{6}}\to R \quad \dfrac{}{p, A \Rightarrow p \ _{7}}\,\text{Id}}{B, A \Rightarrow p \ _{5}}\to L}{A \Rightarrow B \to p \ _{3}}\to R \quad \dfrac{}{q \Rightarrow q \ _{4}}}{A \Rightarrow q \ _{2}}}{\Rightarrow W \ _{1}}\to R$$

... we get a **G3i**-derivation of $\Rightarrow W$.

A proof-search example (10)

Let us go back to the backtrack point in sequent 8
(both A and B can be chosen as main formula of $\to L$)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{p, B, A \stackrel{\mathrm{u}}{\Rightarrow} q \;_{8}}{B, A \stackrel{\mathrm{b}}{\Rightarrow} p \to q \;_{6}} \to R_2
      \qquad
      \dfrac{}{p, A \stackrel{\mathrm{u}}{\Rightarrow} p \;_{7}} \mathrm{Id}
    }{B, A \stackrel{\mathrm{u}}{\Rightarrow} p \;_{5}} \to L
  }{A \stackrel{\mathrm{b}}{\Rightarrow} B \to p \;_{3}} \to R_2
  \qquad
  \dfrac{}{q \stackrel{\mathrm{u}}{\Rightarrow} q \;_{4}}
}{
  \dfrac{A \stackrel{\mathrm{u}}{\Rightarrow} q \;_{2}}{\stackrel{\mathrm{u}}{\Rightarrow} W \;_{1}} \to R_2
} \to L
$$

Let us choose B instead of A

A proof-search example (11)

$$W \;=\; A \to q \qquad A \;=\; (B \to p) \to q \qquad B \;=\; (p \to q) \to p$$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        p, B, A \stackrel{\mathrm{b}}{\Rightarrow} p \to q \;{}_{9} \qquad p, A \stackrel{\mathrm{u}}{\Rightarrow} q \;{}_{10}
      }{
        p, B, A \stackrel{\mathrm{u}}{\Rightarrow} q \;{}_{8}
      } \to L
    }{
      B, A \stackrel{\mathrm{b}}{\Rightarrow} p \to q \;{}_{6}
    } \to R_2
    \qquad
    \cfrac{}{p, A \stackrel{\mathrm{u}}{\Rightarrow} p \;{}_{7}} \mathrm{Id}
  }{
    \cfrac{
      B, A \stackrel{\mathrm{u}}{\Rightarrow} p \;{}_{5}
    }{
      A \stackrel{\mathrm{b}}{\Rightarrow} B \to p \;{}_{3}
    } \to R_2
  } \to L
  \qquad
  \cfrac{}{q \stackrel{\mathrm{u}}{\Rightarrow} q \;{}_{4}}
}{
  \cfrac{
    A \stackrel{\mathrm{u}}{\Rightarrow} q \;{}_{2}
  }{
    \stackrel{\mathrm{u}}{\Rightarrow} W \;{}_{1}
  } \to R_2
}
$$

Sequent 9 is blocked and

$$p, B, A \;\vdash_{\tilde{\mathcal{E}}}\; p$$

We have to apply $\to R_1$ with main formula $p \to q$.

A proof-search example (12)

$$W = A \to q \qquad A = (B \to p) \to q \qquad B = (p \to q) \to p$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, B, A \overset{b}{\Rightarrow} q \;{}_{11}}{p, B, A \overset{b}{\Rightarrow} p \to q \;{}_{9}} \to R_1 \qquad p, A \overset{u}{\Rightarrow} q \;{}_{10}}{p, B, A \overset{u}{\Rightarrow} q \;{}_{8}} \to L}{B, A \overset{b}{\Rightarrow} p \to q \;{}_{6}} \to R_2 \qquad \dfrac{}{p, A \overset{u}{\Rightarrow} p \;{}_{7}}\,\mathrm{Id}}{B, A \overset{u}{\Rightarrow} p \;{}_{5}} \to L}{A \overset{b}{\Rightarrow} B \to p \;{}_{3}} \to R_2 \qquad \dfrac{}{q \overset{u}{\Rightarrow} q \;{}_{4}}}{A \overset{u}{\Rightarrow} q \;{}_{2}}}{\overset{u}{\Rightarrow} W \;{}_{1}} \to R_2$$

Sequent 11 is blocked.

We cannot apply left-rules.
The construction of the derivation fails.

Conclusions

- We have presented **Gbu**, a terminating sequent calculus for **IPL**. **Gbu** is a notational variant of **G3i**, where sequents are labelled to mark the right-focused phase.
- Note that focusing techniques reduce the search space limiting the use of contraction, but they do not guarantee termination of proof-search (see, e.g., the right-focused calculus **LJQ** [Dyckoff&Lengrand,2006]).

  To get this, one has to introduce extra machinery. An efficient solution is loop-checking implemented by history mechanisms

  A. Heuerding et al., *Efficient loop-check for backward proof search in some non-classical propositional logics*, Tableaux 96.

  J. M. Howe., *Two loop detection mechanisms: A comparison*, Tableaux 97.

Conclusions

Here we propose a different approach, based on an evaluation relation defined on sequents.

- Histories

  Require space to store the right formulas already used so to direct and possibly stop the proof-search.

- In our approach

  We have to compute evaluation relations when right-implication is treated.

  With an appropriate implementation of data structures:

  - The evaluation relation $\vdash_{\tilde{\mathcal{E}}}$ can be computed in time linear in the size of the arguments.
  - The overall time needed to compute $\vdash_{\tilde{\mathcal{E}}}$ in the construction of a branch with root σ is $O(|\sigma|^3)$.

M. Ferrari, C. Fiorentini, G.Fiorino. *Simplification Rules for Intuitionistic Propositional Tableaux*. TOCL, 2012.

A comparison with history based calculi

A strict comparison between **Gbu** and history based approach is hard.

We provide an example where **Gbu** outperforms history-based calculi.

Let us search for a derivation of

$$\Gamma^* \Rightarrow \bot \qquad \Gamma^* = \{\, p_1 \to \bot,\ p_2 \to \bot,\ \ldots,\ p_n \to \bot \,\}$$

in

- **MJ**$^{(\dagger)}$ with histories (Swiss style) [Howe, Tableaux 97].
- **Gbu**

(†) **MJ** (alias *LJT*) is the Herbelin sequent calculus isomorphic to Natural Deduction [CSL,1994]

A comparison with history based calculi

Some rules of *MJ*

$\mathcal{H}$: history set

- Left focus

$$\frac{\Gamma \xrightarrow{A} D;\ \mathcal{H}}{\Gamma \Longrightarrow D;\ \mathcal{H}}\ \mathit{focus}$$

$A \in \Gamma$

D is a prop.variable or $\bot$ or a disjunction

- $\bot$ (axiom rule)

$$\frac{}{\Gamma \xrightarrow{\bot} C;\ \mathcal{H}}\ \bot$$

- Left implication

$$\frac{\Gamma \Longrightarrow A;\ C, \mathcal{H} \qquad \Gamma \xrightarrow{B} C;\ \mathcal{H}}{\Gamma \xrightarrow{A \to B} C;\ \mathcal{H}}\ \to_L \quad C \notin \mathcal{H}$$

A comparison with history based calculi

$$
\begin{array}{lcl}
\Gamma^* & = & \{\, p_1 \to \bot,\ p_2 \to \bot,\ \ldots,\ p_n \to \bot \,\} \\
\mathcal{H}_n & = & \{\, \bot,\ p_1, \ldots,\ p_n \,\}
\end{array}
$$

In proof-search, we build the tree

$$
\dfrac{
 \dfrac{
  \dfrac{
   \dfrac{
     \dfrac{\dfrac{\Gamma^* \xrightarrow{p_k \to \bot} p_j;\ \mathcal{H}_n}{\Gamma^* \Longrightarrow p_j;\ \mathcal{H}_n}\,\textit{focus} \quad \cdots}{\Gamma^* \Longrightarrow p_2;\ \{p_1, \bot\}}
     \quad
     \dfrac{}{\Gamma^* \xrightarrow{\bot} p_1;\ \{\bot\}}\,\bot
   }{\Gamma^* \xrightarrow{p_2 \to \bot} p_1;\ \{\bot\}}\,\to L
  }{\Gamma^* \Longrightarrow p_1;\ \{\bot\}}\,\textit{focus}
  \quad
  \dfrac{}{\Gamma^* \xrightarrow{\bot} \bot;\ \emptyset}\,\bot
 }{\Gamma^* \xrightarrow{p_1 \to \bot} \bot;\ \emptyset}\,\to L
}{\Gamma^* \Longrightarrow \bot;\ \emptyset}\,\textit{focus}
$$

The topmost sequent cannot be expanded:
we cannot apply $\to L$ since p_j is already in $\mathcal{H}_n$.

The left-most branch chains $n + 1$ applications of $\to L$.

A comparison with history based calculi

In **Gbu**:

for every $p_j \to \bot$ chosen as main formula of $\to L$,
the generated proof-tree has depth 2.

$$\Gamma^* \; = \; \{\, p_1 \to \bot,\; p_2 \to \bot,\; \ldots,\; p_n \to \bot \,\}$$

$$\dfrac{\Gamma^* \overset{\mathrm{b}}{\Rightarrow} p_j \qquad \dfrac{}{\bot, \Gamma^* \overset{\mathrm{u}}{\Rightarrow} \bot}\,\bot L}{\Gamma^* \overset{\mathrm{u}}{\Rightarrow} \bot}\,\to L$$

We cannot expand the leftmost premise (it is blocked).

Future work

- The evaluation relation $\vdash_{\mathcal{E}}$ only exploits the information in the left-hand side of a sequent.

  We are investigating the use of more expressive evaluation relations to better grasp the information conveyed by a sequent and further reduce the search space (e.g., evaluation relations taking into account also the right formula of a sequent).

- We aim to extend the use of these techniques to other logics having a Kripke semantics.

Implementation

We have implemented **Gbu** using

- JTabWb

  A Java framework for developing provers based on terminating sequent or tableau calculi.

The framework provides support for:

- generation of proof-traces (histories of proof-search);
- LaTeXrendering of proofs;
- countermodel generation.

Available at:

`http://www.dicom.uninsubria.it/~ferram`